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Stochastic Euler equations of fluid dynamics with Lévy noise

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Abstract. In this work we prove the existence and uniqueness of pathwise solutions up to a stopping time to the stochastic Euler equations perturbed by additive and multiplicative Lévy noise in two and three dimensions. The existence of a unique maximal solution is also proved.

Keywords: Euler equations of fluid dynamics, Lévy process, commutator estimates

1. Introduction

The Euler equations are a set of quasilinear partial differential equations which describe the motion of inviscid fluid flow. The mathematical theory for the deterministic Euler equations have been studied by numerous mathematicians in the past several decades ([5,11,13,22,28,45,46], references therein). The global existence and uniqueness of solutions for the Euler equations is still an open problem in three dimensions.

The two-dimensional stochastic Euler equations have been considered by several authors ([2,4,8,10, 14,15,24]). The existence of a martingale solution in a bounded domain is proved in [4] and in a smooth subset of $\mathbb{R}^2$ is proved in [8]. The stochastic Euler equations with periodic boundary conditions are considered in [10] and the existence of solution on Loeb space with prescribed Wiener process is proved using nonstandard analysis. An existence and uniqueness theorem of strong solution is proved in [2] and [24]. By considering Euler equation as the equation of geodesic on the volume preserving diffeomorphism group, the authors in [14] obtained the existence of global solutions to 2-D stochastic Euler equations. The weak existence of an H^1-regular solution with Dirichlet and periodic boundary conditions on bounded domains is obtained in [15]. The papers [25,33,34] considered the stochastic Euler equations in 3 dimensions with additive Gaussian noise and the paper [20] considered multiplicative Gaussian noise. Navier–Stokes equations with Lévy noise is considered in the papers [7,18,19,36,37,42], for example. We consider the stochastic Euler equations in two and three dimensions perturbed by Lévy noise and

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prove the local in time existence and uniqueness of strong solutions generalizing the frequency truncation method used in the deterministic context in [17] and [35], and this is a new technique for stochastic quasilinear PDEs developed here. The use of Fourier-harmonic analysis techniques clarify the abstract treatment of the noise covariance structure and other technical calculations found in the related literature. Moreover, to the best of the authors knowledge, this work appears to be the first in establishing a unique solution to the stochastic Euler equations with jump noise.

The construction of the paper is as follows. In Section 2, we formulate the abstract stochastic incompressible, Euler model of fluid dynamics perturbed by additive Lévy noise. By considering a truncated model in the frequency domain, we prove a-priori energy estimates in $\mathrm{H}^s(\mathbb{R}^n)$, for $s > n/2 + 1$ up to a stopping time (Proposition 2.11). The existence and uniqueness of the local in time strong solution (Theorem 3.11 and Theorem 3.14) is obtained in Section 3 by showing that the family of solutions to the truncated incompressible, stochastic Euler equations is Cauchy. The existence of a unique maximal local strong solution (Theorem 3.16) is also proved in this section. In Section 4, we consider the stochastic Euler equations with multiplicative Lévy noise and discuss the existence and uniqueness of the local strong solution (Theorem 4.4).

2. Stochastic Euler equations

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space. The incompressible stochastic Euler equations (for $n = 2, 3$) are given by

$$\frac{\partial \mathbf{u}(x,t,\omega)}{\partial t} + \big(\mathbf{u}(x,t,\omega)\cdot\nabla\big)\mathbf{u}(x,t,\omega) = -\nabla p(x,t,\omega) + \mathbf{f}(x,t,\omega), \quad \text{in } \mathbb{R}^n \times (0,T)\times\Omega, \tag{2.1}$$

$$\nabla\cdot\mathbf{u}(x,t,\omega) = 0, \quad \text{in } \mathbb{R}^n \times (0,T)\times\Omega, \tag{2.2}$$

$$\mathbf{u}(x,0,\omega) = \mathbf{u}_0(x,\omega), \quad \text{in } \mathbb{R}^n \times \Omega, \tag{2.3}$$

where $\mathbf{u}(x,t,\omega)$ is the velocity field $p(x,t,\omega)$ is the pressure field and $\mathbf{f}(x,t,\omega)$ is the external random forcing. The equations are defined on the whole space $\mathbb{R}^n$, $n = 2$ or 3 with the initial data

$$\mathbf{u}_0 \in \mathrm{L}^2\big(\Omega; \mathrm{H}^s\big(\mathbb{R}^n\big)\big),$$

satisfying $\nabla\cdot\mathbf{u}_0 = 0$ for $s > n/2 + 1$. Here $\mathrm{H}^s(\mathbb{R}^n)$ ($\equiv (\mathrm{H}^s(\mathbb{R}^n))^n$) is the Hilbertian Sobolev space of order s. Let us define the divergence free space $\mathcal{H}$ by

$$\mathcal{H} := \big\{\mathbf{u}\in\mathrm{L}^2\big(\mathbb{R}^n\big)\big|\nabla\cdot\mathbf{u} = 0\big\}.$$

In order to eliminate the pressure, we project the equations onto the space of divergence free functions on $\mathcal{H}$, by taking the Helmholtz–Hodge orthogonal projection $\mathrm{P}_{\mathcal{H}}$ from L^2 onto $\mathcal{H}$. Let us now express the projection operator $\mathrm{P}_{\mathcal{H}} : \mathrm{L}^2 \to \mathcal{H}$ in terms of the Riesz transform. Let us define the Riesz transform by

$$\mathrm{R}_j = -\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}, \quad j = 1,\dots,n \quad \text{so that} \quad \mathrm{R}_j\mathrm{R}_k = \frac{\partial^2}{\partial x_j\,\partial x_k}(-\Delta)^{-1}, \quad j,k = 1,\dots,n.$$

For $f \in \mathrm{L}^2(\mathbb{R}^n)$ and $j = 1, \dots, n$, the Fourier transform of the Riesz transform $\mathrm{R}_j : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is $\widehat{\mathrm{R}_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{f}(\xi)$. Hence, we have $\sum_{j=1}^n \mathrm{R}_j^2 = -\mathrm{I}$ (see [9]). The Riesz transform can also be defined in terms of the singular integral operators (Example 2, page 4, [12], Chapter III, [44]). For $f \in \mathrm{L}^2(\mathbb{R}^n)$, set

$$\mathrm{R}_j f(x) = C_n \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\backslash \mathrm{B}_\varepsilon(0)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,\mathrm{d}y,$$

where $\mathrm{B}_\varepsilon(0)$ is a ball centered at the origin of radius ε and $C_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$ is the volume of unit ball in $\mathbb{R}^n$. For any $\mathbf{u}(x) = (u_1(x), \dots, u_n(x))$, one can deduce the Helmholtz–Hodge orthogonal projection operator $\mathrm{P}_{\mathcal{H}}$ as

$$(\mathrm{P}_{\mathcal{H}}u)_j(x) = \sum_{k=1}^n (\delta_{jk} + \mathrm{R}_j\mathrm{R}_k)u_k(x), \quad j = 1, \dots, n.$$

Equivalently, by making use of the Fourier transform, we find

$$\widehat{(\mathrm{P}_{\mathcal{H}}u)}_j(\xi) = \sum_{k=1}^n \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{u}_k(\xi), \quad j = 1, \dots, n.$$

Hence, $\mathrm{P}_{\mathcal{H}}$ is an orthogonal projection onto the kernel of the divergence operator so that $\mathrm{P}_{\mathcal{H}}(\nabla p) = 0$ and is a pseudodifferential operator of order 0 and belongs to the class $\Psi^0_{1,0}$ ([21]). By using the Fourier transform, it can be seen that the operators $\mathrm{J}^s := (\mathrm{I} - \Delta)^{s/2}$ and $\mathrm{P}_{\mathcal{H}}$ commute:

$$\begin{aligned}
\widehat{\mathrm{J}^s\mathrm{P}_{\mathcal{H}}u}_j(\xi) &= \left(1 + |\xi|^2\right)^{s/2}\widehat{\mathrm{P}_{\mathcal{H}}u}_j(\xi) = \left(1 + |\xi|^2\right)^{s/2}\sum_{k=1}^n \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{u}_k(\xi) \\
&= \sum_{k=1}^n \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\left(1 + |\xi|^2\right)^{s/2}\widehat{u}_k(\xi) = \sum_{k=1}^n \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{\mathrm{J}^s u}_k(\xi) \\
&= \widehat{\mathrm{P}_{\mathcal{H}}\mathrm{J}^s u}_j(\xi),
\end{aligned} \tag{2.4}$$

for all $\xi \in \mathbb{R}^n$ and hence $\mathrm{J}^s\mathrm{P}_{\mathcal{H}} = \mathrm{P}_{\mathcal{H}}\mathrm{J}^s$. Let us take the Helmholtz–Hodge orthogonal projection $\mathrm{P}_{\mathcal{H}}$ on (2.1) to get

$$\frac{\partial \mathbf{u}(x,t,\omega)}{\partial t} = -\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}(x,t,\omega)\cdot\nabla\big)\mathbf{u}(x,t,\omega) + \mathrm{P}_{\mathcal{H}}\mathbf{f}(x,t,\omega), \quad \text{in } \mathbb{R}^n \times (0,T) \times \Omega, \tag{2.5}$$

$$\mathbf{u}(x,0,\omega) = \mathbf{u}_0(x,\omega), \quad \text{in } \mathbb{R}^n \times \Omega. \tag{2.6}$$

Then the stochastic Euler equations perturbed by additive Lévy noise can be written as an Itô stochastic differential equation in $(0,T)$ after taking $\mathrm{P}_{\mathcal{H}}$ as

$$\mathrm{d}\mathbf{u}(x,t) = -\mathrm{P}_{\mathcal{H}}\big[\big(\mathbf{u}(x,t)\cdot\nabla\big)\mathbf{u}(x,t)\big]\,\mathrm{d}t + \Phi\,\mathrm{dW}(x,t) + \int_{\mathrm{Z}} \gamma(t-,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z), \tag{2.7}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \tag{2.8}$$

where $\mathbf{u}_0 \in \mathrm{L}^2(\Omega; \mathrm{H}^s(\mathbb{R}^n))$ for $s > n/2 + 1$ with $\nabla \cdot \mathbf{u}_0 = 0$ and Z is a measurable space (where the solution has its paths) such that $\mathrm{Z} \in \mathscr{B}(\mathcal{H}\backslash\{0\})$. In (2.7), Φ is an operator having the properties discussed below and $\mathrm{W}(\cdot, \cdot)$ is a cylindrical Wiener process defined on $\mathcal{H}$. The operator Φ has the following properties:

(i) The operator $\Phi \in \mathscr{L}(\mathcal{H}, \mathrm{H}^s)$, with $\mathrm{Tr}_{\mathcal{H}}(\Phi^*\Phi) < \infty$,
(ii) $\mathrm{Tr}_{\mathcal{H}}((\mathrm{J}^s\Phi)^*\mathrm{J}^s\Phi) < \infty$.

Here $\mathscr{L}(\mathcal{H}, \mathrm{H}^s)$ denotes the space of all bounded linear operators from $\mathcal{H}$ to H^s. For an orthonormal basis $\{e_j(x)\}_{j=1}^{\infty}$ in $\mathcal{H}$, $\mathrm{W}(\cdot, \cdot)$ can be written as $\mathrm{W}(x, t) = \sum_{j=1}^{\infty} e_j(x)\beta_j(t)$, where $\beta_j(\cdot)$'s are a sequence of one-dimensional Brownian motions in $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\lambda(\mathrm{d}z)$ be a σ-finite Lévy measure on $\mathcal{H}$ with an associated Poisson random measure $\mathcal{N}(\mathrm{d}t, \mathrm{d}z)$. Let $\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}z) := \mathcal{N}(\mathrm{d}t, \mathrm{d}z) - \lambda(\mathrm{d}z)\,\mathrm{d}t$ be the compensated Poisson random measure, that is, $\mathbb{E}(\mathcal{N}(\mathrm{d}t, \mathrm{d}z)) = \lambda(\mathrm{d}z)\,\mathrm{d}t$ (Theorem 35, Section 4, [39]). The jump coefficient $\gamma : [0, T] \times \mathrm{Z} \to \mathrm{H}^s \cap \mathcal{H}$ is an $\mathrm{H}^s \cap \mathcal{H}$-valued function such that $\int_0^T \int_{\mathrm{Z}} \|\gamma(t, z)\|_{\mathrm{H}^s}^2 \lambda(\mathrm{d}z)\,\mathrm{d}t < \infty$, for any $T > 0$. The processes $\mathrm{W}(\cdot, \cdot)$ and $\widetilde{\mathcal{N}}(\cdot, \cdot)$ are mutually independent.

Example 2.1 (An example of the operator Φ)**.** The symbol class $\sum^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, consists of the C^∞-functions $\mathrm{a}(\cdot)$ on $\mathbb{R}^n$ such that for any multi-index α, there exists a constant C_α with

$$\left|\partial^\alpha \mathrm{a}(x)\right| \leqslant C_\alpha\left(1 + |x|^2\right)^{\frac{m-|\alpha|}{2}},$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We shall say that $\mathrm{a} \in \sum^m(\mathbb{R}^n)$ is of order m.

A symbol $\mathrm{a} \in \sum^m(\mathbb{R}^n \times \mathbb{R}^n)$ define an operator, denoted by $\mathcal{P}_{\mathrm{a}}$, by the formula (Section 18.5, [21])

$$(\mathcal{P}_{\mathrm{a}}u)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\mathrm{a}(x, \xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, u \in C_c^\infty\left(\mathbb{R}^n\right).$$

Let $\mathrm{a} \in \sum^m(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol. A *pseudodifferential operator with Weyl symbol* a is defined by the oscillating integral

$$(\mathcal{P}_{\mathrm{a}}u)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\mathrm{a}\left(\frac{x+y}{2}, \xi\right)u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad u \in C_c^\infty\left(\mathbb{R}^n\right).$$

An operator $\mathcal{P}_{\mathrm{a}}$ with Weyl symbol a is self-adjoint if and only if a is a real function ([21]). We can write $(\mathcal{P}_{\mathrm{a}}u)(x)$ as

$$(\mathcal{P}_{\mathrm{a}}u)(x) = \int_{\mathbb{R}^n} \mathrm{K}(x, y)u(y)\,\mathrm{d}y, \quad \text{where } \mathrm{K}(x, y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\mathrm{a}\left(\frac{x+y}{2}, \xi\right)\mathrm{d}\xi,$$

and the $\mathrm{Tr}(\mathcal{P}_{\mathrm{a}})$ is given by

$$\mathrm{Tr}(\mathcal{P}_{\mathrm{a}}) = \int_{\mathbb{R}^n} \mathrm{K}(x, x)\,\mathrm{d}x = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{a}(x, \xi)\,\mathrm{d}\xi\,\mathrm{d}x.$$

If $m < -2n$ ($m = -k$ with $k > 0$), then $\mathcal{P}_\mathrm{a}$ extends to a trace class operator on $\mathrm{H}^s(\mathbb{R}^n)$ (Proposition 27.2, [43]) and $\mathrm{Tr}(\mathcal{P}_\mathrm{a}) < \infty$, since

$$\begin{aligned}
\mathrm{Tr}(\mathcal{P}_\mathrm{a}) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathrm{a}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \leqslant \frac{C_0}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+|x|^2+|\xi|^2\right)^{\frac{m}{2}}\mathrm{d}\xi\,\mathrm{d}x \\
&= \frac{C_0}{(2\pi)^n}\int_{\mathbb{R}^{2n}}\frac{\mathrm{d}y}{(1+|y|^2)^{\frac{k}{2}}} = \frac{C_0}{2^{n-1}\Gamma(n)}\int_0^\infty \frac{r^{2n-1}}{(1+r^2)^{\frac{k}{2}}}\,\mathrm{d}r = \frac{C_0\Gamma(\frac{k}{2}-n)}{\Gamma(\frac{k}{2})} < \infty,
\end{aligned} \tag{2.9}$$

if $k > 2n$, i.e., $m < -2n$. It can be easily seen that

$$\left|\partial^\alpha\left[\left(1+|\xi|^2\right)^{s/2}\mathrm{a}(x,\xi)\right]\right| \leqslant C_\alpha\left(1+|x|^2+|\xi|^2\right)^{\frac{m+s-|\alpha|}{2}}$$

and hence $(1+|\xi|^2)^{s/2}\mathrm{a}(x,\xi) \in \sum^{m+s}(\mathbb{R}^n\times\mathbb{R}^n)$. Also if $-m > 2s$ and $m+2s < -2n$, then $\mathrm{Tr}(\mathrm{J}^{2s}\mathcal{P}_\mathrm{a}) < \infty$. We can take $\Phi^*\Phi = \mathcal{P}_\mathrm{a}$ with $\mathrm{a} \in \sum^m(\mathbb{R}^n\times\mathbb{R}^n)$, for $m < -2n$ so that $\mathrm{Tr}(\mathcal{P}_\mathrm{a}) < \infty$. In addition, if $m+2s < -2n$, then $\mathrm{Tr}((\mathrm{J}^s\Phi)^*\mathrm{J}^s\Phi) = \mathrm{Tr}(\mathrm{J}^{2s}\mathcal{P}_\mathrm{a}) < \infty$.

From now onwards, we use the notation Tr for $\mathrm{Tr}_\mathcal{H}$. We also use $K_1 = \mathrm{Tr}(\Phi\Phi^*)$, $K_2 = \mathrm{Tr}(\mathrm{J}^s\Phi(\mathrm{J}^s\Phi)^*)$, $K_3 := K_3(T) = \int_0^T\int_\mathrm{Z}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t$ and $K_4 := K_4(T) = \int_0^T\int_\mathrm{Z}\|\gamma(t,z)\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}t$, for any $T > 0$ in the rest of the paper. Let us recall the commutator estimates of Kato and Ponce [23] used in this paper.

Lemma 2.2 (Lemma XI, [23])**.** *If $s \geqslant 0$ and $1 < p < \infty$, then*

$$\left\|\mathrm{J}^s(fg) - f\left(\mathrm{J}^s g\right)\right\|_{\mathrm{L}^p} \leqslant C_p\left(\|\nabla f\|_{\mathrm{L}^\infty}\left\|\mathrm{J}^{s-1}g\right\|_{\mathrm{L}^p} + \left\|\mathrm{J}^s f\right\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^\infty}\right). \tag{2.10}$$

Remark 2.3. From (2.10), it can be easily seen that for $s \geqslant 0$ and $1 < p < \infty$, the nonlinear term satisfy the estimate:

$$\left\|\mathrm{J}^s\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] - (\mathbf{u}\cdot\nabla)\left(\mathrm{J}^s\mathbf{u}\right)\right\|_{\mathrm{L}^p} \leqslant C_p\left(\|\nabla\mathbf{u}\|_{\mathrm{L}^\infty}\left\|\mathrm{J}^{s-1}\nabla\mathbf{u}\right\|_{\mathrm{L}^p} + \left\|\mathrm{J}^s\mathbf{u}\right\|_{\mathrm{L}^p}\|\nabla\mathbf{u}\|_{\mathrm{L}^\infty}\right). \tag{2.11}$$

If $\mathbf{u}$ is divergence free, then we have

$$\left((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{v}\right)_{\mathrm{L}^2} = \sum_{i,k=1}^n\int_{\mathbb{R}^n}u_i\frac{\partial v_k}{\partial x_j}v_k\,\mathrm{d}x = \frac{1}{2}\sum_{i,k=1}^n\int_{\mathbb{R}^n}u_i\frac{\partial v_k^2}{\partial x_j}\,\mathrm{d}x = -\frac{1}{2}\sum_{i,k=1}^n\int_{\mathbb{R}^n}\frac{\partial u_i}{\partial x_j}v_k^2\,\mathrm{d}x = 0, \tag{2.12}$$

for all $\mathbf{u},\mathbf{v} \in \mathrm{H}^s(\mathbb{R}^n)$, $s \geqslant 0$. Thus it is immediate that

$$\left((\mathbf{u}\cdot\nabla)\left(\mathrm{J}^s\mathbf{u}\right),\mathrm{J}^s\mathbf{u}\right)_{\mathrm{L}^2} = 0.$$

In Remark 2.3, if we take $p = 2$, then we get the following corollary:

Corollary 2.4. *For $s \geqslant 0$, there exists a constant $C = C(n,s)$ such that, for all $\mathbf{u} \in \mathrm{H}^s(\mathbb{R}^n)$, $s \geqslant 0$, and $\nabla\cdot\mathbf{u} = 0$, we have*

$$\left|\left(\mathrm{J}^s\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right],\mathrm{J}^s\mathbf{u}\right)_{\mathrm{L}^2}\right| \leqslant C\|\nabla\mathbf{u}\|_{\mathrm{L}^\infty}\|\mathbf{u}\|_{\mathrm{H}^s}^2. \tag{2.13}$$

Let us now define the notion of local and maximal strong solutions of the stochastic Euler equations with Lévy noise.

Definition 2.5 (Local strong solution)**.** We say that the pair $(\mathbf{u},\tau)$ is a *local strong (pathwise) solution* for the stochastic Euler equations (2.7)–(2.8) if

(i) the symbol τ is a strictly positive stopping time, i.e., $\mathbb{P}(\tau>0)=1$,
(ii) for $t>0$, the symbol $\mathbf{u}$ denotes progressively measurable stochastic process such that
  (a) $\mathbf{u}(\cdot)\in \mathrm{L}^2(\Omega;\mathrm{D}(0,t;\mathrm{H}^s(\mathbb{R}^n)))$, for $s>n/2+1$ with $\mathbf{u}(\cdot)=\mathbf{u}(\cdot\wedge\tau)$, where $\mathrm{D}(0,t;\mathrm{H}^s(\mathbb{R}^n))$ is the space of all càdlàg paths from $[0,t)$ to $\mathrm{H}^s(\mathbb{R}^n)$,
  (b) $\mathbf{u}(\cdot)$ satisfies

$$\mathbf{u}(t\wedge\tau)=\mathbf{u}_0-\int_0^{t\wedge\tau}\mathrm{P}_{\mathcal{H}}\big[(\mathbf{u}\cdot\nabla)\mathbf{u}\big](s)\,\mathrm{d}s+\int_0^{t\wedge\tau}\Phi\,\mathrm{dW}(s)+\int_0^{t\wedge\tau}\int_{\mathrm{Z}}\gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z),$$

  (c) $\mathbf{u}(\cdot)$ satisfies the energy estimate

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant t\wedge\tau}\big\|\mathbf{u}(s)\big\|_{\mathrm{H}^s}^2\Big]<\infty. \tag{2.14}$$

Definition 2.6 (Maximal local strong solution)**.** Let $\mathbf{u}(\cdot)$ be a predictable process and τ_∞ be a strictly positive stopping time. The pair $(\mathbf{u},\tau_\infty)$ is said to be a *maximal local strong (pathwise) solution* for the stochastic Euler equations (2.7)–(2.8), if there exists an increasing sequence τ_n with

$$\tau_n\uparrow\tau_\infty\quad \text{a.s.},$$

such that the pair $(\mathbf{u},\tau_n)$ is a local strong solution to (2.7)–(2.8) so that

$$\sup_{0\leqslant s\leqslant\tau_\infty}\big\|\mathbf{u}(s)\big\|_{\mathrm{H}^s}=\infty$$

on the set $\{\omega\in\Omega:\tau_\infty(\omega)<\infty\}$.

We are now ready to state the main theorem of our paper.

Theorem 2.7. *Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant0},\mathbb{P})$ be a given filtered probability space. Let τ_N be the stopping time defined by*

$$\tau_N:=\inf_{t\geqslant0}\big\{t:\big\|\nabla\mathbf{u}(t)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{u}(t)\big\|_{\mathrm{H}^{s-1}}\geqslant N\big\}, \tag{2.15}$$

and let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathrm{H}^s(\mathbb{R}^n))$, for $s>n/2+1$ be given. Then there exists a local in time strong solution $(\mathbf{u},T\wedge\tau_N)$ of the stochastic incompressible Euler equations with Lévy noise ((2.7)–(2.8)) such that, for any $T>0$

(i) *the energy estimate*

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T\wedge\tau_N}\|\mathbf{u}(s)\|_{\mathrm{H}^s}^2\Big]\leqslant\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+K_2T+K_4\big)e^{4NT}<\infty,$$

where $K_2=\mathrm{Tr}((\mathrm{J}^s\Phi)^*\mathrm{J}^s\Phi)$ *and* $K_4=\int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}t$*,*

(ii) *for a given* $0<\delta<1$*,*

$$\mathbb{P}\{\tau_N>\delta\}\geqslant 1-C\delta^2\big\{2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18\big(K_2\delta+K_4(\delta)\big)\big\},$$

where C *is a positive constant independent of* $\mathbf{u}$ *and* δ*,*

(iii) $\mathbf{u}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^s(\mathbb{R}^n)))$*,*

(iv) *the* $(\mathscr{F}_t)_{t\in[0,T]}$*-adapted paths of* $(\mathbf{u},T\wedge\tau_N)$ *are càdlàg,*

(v) *the solution* $(\mathbf{u},T\wedge\tau_N)$ *is pathwise unique,*

(vi) *there exists a unique maximal local strong solution* $(\mathbf{u},\tau_\infty)$*, where* $\tau_\infty=\lim_{N\to\infty}\tau_N$*.*

2.1. The truncated stochastic Euler equations

Let us define the Fourier truncation $\mathcal{S}_R$ ([17]) as follows:

$$\widehat{\mathcal{S}_R f}(\xi)=\mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{f}(\xi),$$

where B_R, a ball of radius R centered at the origin and $\mathbf{1}_{\mathrm{B}_R}(\cdot)$ is the indicator function. For $s\geqslant 0$, we have

$$\|\mathcal{S}_R f\|_{\mathrm{H}^s}\leqslant C\|f\|_{\mathrm{H}^s},\tag{2.16}$$

$$\|\mathcal{S}_R f-f\|_{\mathrm{H}^s}\leqslant C\left(\frac{1}{R}\right)^k\|f\|_{\mathrm{H}^{s+k}},\tag{2.17}$$

$$\big\|(\mathcal{S}_R-\mathcal{S}_{R'})f\big\|_{\mathrm{H}^s}\leqslant C\max\left\{\left(\frac{1}{R}\right)^k,\left(\frac{1}{R'}\right)^k\right\}\|f\|_{\mathrm{H}^{s+k}},\tag{2.18}$$

where C is a generic constant independent of R.

Let us consider the truncated (in the frequency domain with cut off $\mathcal{S}_R$) stochastic Euler equations in the whole space $\mathbb{R}^n$ as

$$\frac{\partial\mathbf{u}^R(x,t,\omega)}{\partial t}=-\mathcal{S}_R\big(\mathbf{u}^R(x,t,\omega)\cdot\nabla\big)\mathbf{u}^R(x,t,\omega)-\nabla p^R(x,t,\omega)+\mathcal{S}_R\mathrm{f}(x,t,\omega),\tag{2.19}$$

$$\nabla\cdot\mathbf{u}^R(x,t,\omega)=0,\tag{2.20}$$

$$\mathbf{u}^R(x,0,\omega)=\mathcal{S}_R\mathbf{u}_0(x,\omega),\tag{2.21}$$

for $(x,t,\omega)\in\mathbb{R}^n\times(0,T)\times\Omega$. The divergence free condition for $\mathbf{u}^R$ can be obtained easily as

$$\widehat{\nabla\cdot\mathbf{u}^R}(\xi)=i\xi\cdot\widehat{\mathbf{u}^R}(\xi)=i\xi\cdot\mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{\mathbf{u}}(\xi)=\mathbf{1}_{\mathrm{B}_R}(\xi)i\xi\cdot\widehat{\mathbf{u}}(\xi)=\mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{\nabla\cdot\mathbf{u}}(\xi)=0,\tag{2.22}$$

and hence $\nabla \cdot \mathbf{u}^R = 0$. The cut off operator $\mathcal{S}_R$ and Helmholtz–Hodge orthogonal projection $\mathrm{P}_{\mathcal{H}}$ commutes, since

$$\begin{aligned}\mathrm{P}_{\mathcal{H}}\widehat{\mathcal{S}_R u}_j(\xi) &= \sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{\mathcal{S}_R u}_k(\xi) = \sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{u}_k(\xi) \\ &= \mathbf{1}_{\mathrm{B}_R}(\xi)\sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{u}_k(\xi) = \mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{\mathrm{P}_{\mathcal{H}}u}_j(\xi) = \widehat{\mathcal{S}_R\mathrm{P}_{\mathcal{H}}u}_j(\xi)\end{aligned} \tag{2.23}$$

and hence $\mathrm{P}_{\mathcal{H}}\mathcal{S}_R = \mathcal{S}_R\mathrm{P}_{\mathcal{H}}$. On taking the Helmholtz–Hodge orthogonal projection, we get $\mathrm{P}_{\mathcal{H}}(\nabla p^R) = 0$, since

$$\begin{aligned}\mathrm{P}_{\mathcal{H}}\widehat{\left(\nabla p^R\right)}_j(\xi) &= \sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{\left(\nabla p^R\right)}_k(\xi) = \sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)i\xi_k\widehat{p^R}(\xi) \\ &= \sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)i\xi_k\mathbf{1}_{\mathrm{B}_R}(\xi)\widehat{p}(\xi) = \mathbf{1}_{\mathrm{B}_R}(\xi)\left[\sum_{k=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)i\xi_k\widehat{p}(\xi)\right] \\ &= \mathcal{S}_R\widehat{\mathrm{P}_{\mathcal{H}}(\nabla p)}_j(\xi) = 0.\end{aligned} \tag{2.24}$$

Let us consider the truncated stochastic Euler equations in the Itô stochastic differential form in $(0, T)$ after taking $\mathrm{P}_{\mathcal{H}}$ as

$$\mathrm{d}\mathbf{u}^R = -\mathcal{S}_R\mathrm{P}_{\mathcal{H}}\left(\mathbf{u}^R \cdot \nabla\right)\mathbf{u}^R\,\mathrm{d}t + \sum_{j=1}^{\infty}\mathcal{S}_R\Phi e_j\,\mathrm{d}\beta_j(t) + \int_{\mathrm{Z}}\mathcal{S}_R\gamma(t-, z)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}z), \tag{2.25}$$

$$\mathbf{u}^R(0) = \mathcal{S}_R\mathbf{u}_0. \tag{2.26}$$

By taking a truncated initial data, we ensure that $\mathbf{u}^R$ lie in the space

$$\mathcal{H}_R := \left\{f \in \mathrm{L}^2\left(\mathbb{R}^n\right) : \widehat{f} \text{ is supported in } \mathrm{B}_R, \nabla \cdot f = 0\right\}.$$

Note that $\mathcal{S}_R\mathbf{u}^R = \mathbf{u}^R$ for $\mathbf{u}^R \in \mathcal{H}_R$. Also, by using (2.12) (see (2.22) also) and Hölder's inequality, we obtain

$$\begin{aligned}&\left|\left(\mathcal{S}_R\left[\left(\mathbf{u}_1^R \cdot \nabla\right)\mathbf{u}_1^R\right] - \mathcal{S}_R\left[\left(\mathbf{u}_2^R \cdot \nabla\right)\mathbf{u}_2^R\right], \mathbf{u}_1^R - \mathbf{u}_2^R\right)_{\mathrm{L}^2}\right| \\ &\quad = \left|\left(\left(\left(\mathbf{u}_1^R - \mathbf{u}_2^R\right) \cdot \nabla\right)\mathbf{u}_1^R - \left(\mathbf{u}_2^R \cdot \nabla\right)\left(\mathbf{u}_1^R - \mathbf{u}_2^R\right), \mathcal{S}_R\left(\mathbf{u}_1^R - \mathbf{u}_2^R\right)\right)_{\mathrm{L}^2}\right| \\ &\quad \leqslant \left|\left(\left(\left(\mathbf{u}_1^R - \mathbf{u}_2^R\right) \cdot \nabla\right)\mathbf{u}_1^R, \mathbf{u}_1^R - \mathbf{u}_2^R\right)_{\mathrm{L}^2}\right| + \left|\left(\left(\mathbf{u}_2^R \cdot \nabla\right)\left(\mathbf{u}_1^R - \mathbf{u}_2^R\right), \mathbf{u}_1^R - \mathbf{u}_2^R\right)_{\mathrm{L}^2}\right| \\ &\quad \leqslant \left\|\mathbf{u}_1^R - \mathbf{u}_2^R\right\|_{\mathrm{L}^2}^2\left\|\nabla\mathbf{u}_1^R\right\|_{\mathrm{L}^\infty} \leqslant C\left\|\mathbf{u}_1^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_1^R - \mathbf{u}_2^R\right\|_{\mathrm{L}^2}^2,\end{aligned} \tag{2.27}$$

for $s > n/2 + 1$. Thus, we have

$$\left\|\mathcal{S}_R\left[\left(\mathbf{u}_1^R \cdot \nabla\right)\mathbf{u}_1^R\right] - \mathcal{S}_R\left[\left(\mathbf{u}_2^R \cdot \nabla\right)\mathbf{u}_2^R\right]\right\|_{\mathrm{L}^2} \leqslant C\left\|\mathbf{u}_1^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_1^R - \mathbf{u}_2^R\right\|_{\mathrm{L}^2}, \tag{2.28}$$

and hence $\mathcal{S}_R(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R$ is locally Lipschitz in $\mathcal{H}_R$ whenever $\mathbf{u}^R\in\mathrm{H}^s(\mathbb{R}^n)$, for $s>n/2+1$. Also, by using (2.16) and the algebra property of H^{s-1}-norm, we have

$$\big\|\mathcal{S}_R\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{L}^2}\leqslant\big\|\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{H}^{s-1}}\leqslant\big\|\mathbf{u}^R\big\|_{\mathrm{H}^{s-1}}\big\|\nabla\mathbf{u}^R\big\|_{\mathrm{H}^{s-1}}\leqslant C\big\|\mathbf{u}^R\big\|_{\mathrm{H}^s}^2. \tag{2.29}$$

By using (2.28), (2.29) and Theorem 4.9, [29], there exists a pathwise unique strong solution of problem (2.25)–(2.26) in $\mathrm{L}^2(\Omega;\mathrm{D}(0,\widetilde{T};\mathcal{H}_R))$, where $\widetilde{T}$ depends on R and N (N is defined in (2.67) below). The solution will exist as long as $\|\mathbf{u}^R\|_{\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,\widetilde{T};\mathrm{H}^s(\mathbb{R}^n)))}$ remains finite.

We define a $C_0^\infty(\mathbb{R})$ function $\psi_N(\cdot)$, for each integer N, as ([25])

$$\psi_N(k)=\begin{cases}1, & \text{for all } k\leqslant N,\\ 0, & \text{for all } k\geqslant N+1.\end{cases} \tag{2.30}$$

Let us consider the truncated stochastic Euler equations with the cut off $\psi_N(\cdot)$ (denoting $k^R=\|\nabla\mathbf{u}^R\|_{\mathrm{L}^\infty}+\|\mathbf{u}^R\|_{\mathrm{H}^{s-1}}$) as

$$\mathrm{d}\mathbf{u}^R=-\psi_N\big(k^R\big)\mathcal{S}_R\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\,\mathrm{d}t+\sum_{j=1}^\infty\mathcal{S}_R\Phi e_j\,\mathrm{d}\beta_j(t)+\int_{\mathrm{Z}}\mathcal{S}_R\gamma(t-,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z), \tag{2.31}$$

$$\mathbf{u}^R(0)=\mathcal{S}_R\mathbf{u}_0. \tag{2.32}$$

Note that the presence of the cut off function $\psi_N(\cdot)$ makes the drift term bounded.

Proposition 2.8. *Let $\mathbf{u}^R\in\mathrm{H}^s(\mathbb{R}^n)$, for $s>n/2+1$. Then the nonlinear operator*

$$\mathrm{F}\big(\mathbf{u}^R\big):=\psi_N\big(k^R\big)\mathcal{S}_R\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big]$$

satisfies

$$\big\|\mathrm{F}\big(\mathbf{u}_1^R\big)-\mathrm{F}\big(\mathbf{u}_2^R\big)\big\|_{\mathrm{L}^2}\leqslant\big(C_N+C\big(\big\|\mathbf{u}_1^R\big\|_{\mathrm{H}^s}^2+\big\|\mathbf{u}_2^R\big\|_{\mathrm{L}^2}^2\big)\big)\big\|\mathbf{u}_1^R-\mathbf{u}_2^R\big\|_{\mathrm{H}^s}. \tag{2.33}$$

Proof. Let $\mathbf{u}_i^R\in\mathrm{H}^s(\mathbb{R}^n)$, for $i=1,2$ and for $s>n/2+1$. Then for proving (2.33), we use integration by parts, (2.12) and Hölder's inequality to the term $(\mathrm{F}(\mathbf{u}_1^R)-\mathrm{F}(\mathbf{u}_2^R),\mathbf{u}_1^R-\mathbf{u}_2^R)_{\mathrm{L}^2}$ to get

$$\begin{aligned}
&\big|\big(\mathrm{F}\big(\mathbf{u}_1^R\big)-\mathrm{F}\big(\mathbf{u}_2^R\big),\mathbf{u}_1^R-\mathbf{u}_2^R\big)_{\mathrm{L}^2}\big|\\
&\quad=\big|\big(\psi_N\big(k_1^R\big)\big[\big(\mathbf{u}_1^R\cdot\nabla\big)\mathbf{u}_1^R\big]-\psi_N\big(k_2^R\big)\big[\big(\mathbf{u}_2^R\cdot\nabla\big)\mathbf{u}_2^R\big],\mathcal{S}_R\big(\mathbf{u}_1^R-\mathbf{u}_2^R\big)\big)_{\mathrm{L}^2}\big|\\
&\quad\leqslant\big|\big(\psi_N\big(k_1^R\big)\big(\big(\mathbf{u}_1^R-\mathbf{u}_2^R\big)\cdot\nabla\big)\mathbf{u}_1^R,\mathbf{u}_1^R-\mathbf{u}_2^R\big)_{\mathrm{L}^2}\big|\\
&\qquad+\big|\big(\big(\psi_N\big(k_1^R\big)-\psi_N\big(k_2^R\big)\big)\big(\mathbf{u}_2^R\cdot\nabla\big)\mathbf{u}_1^R,\mathbf{u}_1^R-\mathbf{u}_2^R\big)_{\mathrm{L}^2}\big|\\
&\qquad+\big|\big(\psi_N\big(k_2^R\big)\big(\mathbf{u}_2^R\cdot\nabla\big)\big(\mathbf{u}_1^R-\mathbf{u}_2^R\big),\mathbf{u}_1^R-\mathbf{u}_2^R\big)_{\mathrm{L}^2}\big|\\
&\quad\leqslant\big|\psi_N\big(k_1^R\big)\big|\big\|\nabla\mathbf{u}_1^R\big\|_{\mathrm{L}^\infty}\big\|\mathbf{u}_1^R-\mathbf{u}_2^R\big\|_{\mathrm{L}^2}^2\\
&\qquad+C\big(\big\|\nabla\big(\mathbf{u}_1^R-\mathbf{u}_2^R\big)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{u}_1^R-\mathbf{u}_2^R\big\|_{\mathrm{H}^{s-1}}\big)\big\|\mathbf{u}_2^R\big\|_{\mathrm{L}^2}\big\|\nabla\mathbf{u}_1^R\big\|_{\mathrm{L}^\infty}\big\|\mathbf{u}_1^R-\mathbf{u}_2^R\big\|_{\mathrm{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_N\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{L}^2}+2C\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_1^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_2^R\right\|_{\mathrm{L}^2}\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{L}^2}\\
&\leqslant \left(C_N+C\left(\left\|\mathbf{u}_1^R\right\|_{\mathrm{H}^s}^2+\left\|\mathbf{u}_2^R\right\|_{\mathrm{L}^2}^2\right)\right)\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{H}^s}\left\|\mathbf{u}_1^R-\mathbf{u}_2^R\right\|_{\mathrm{L}^2},
\end{aligned}
\tag{2.34}
$$

where in the third step we used the fact that $(\psi_N(k_2^R)(\mathbf{u}_2^R\cdot\nabla)(\mathbf{u}_1^R-\mathbf{u}_2^R),\mathbf{u}_1^R-\mathbf{u}_2^R)_{\mathrm{L}^2}=0$. Thus the estimate (2.34) implies (2.33). □

2.2. *Energy estimates*

Let us first prove the L^2-energy estimate for the stochastic Euler equations (2.25)–(2.26), which is the truncated stochastic Euler equation without the cut off function $\psi_N(\cdot)$.

Proposition 2.9 (L^2-energy estimate)**.** *Given the initial data $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathrm{L}^2(\mathbb{R}^n))$ with $\nabla\cdot\mathbf{u}_0=0$ be $\mathscr{F}_0$-measurable, then we have*

$$
\mathbb{E}\left[\left\|\mathbf{u}^R(t)\right\|_{\mathrm{L}^2}^2\right]\leqslant\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\right]+t\,\mathrm{Tr}\left(\Phi^*\Phi\right)+\int_0^t\int_{\mathrm{Z}}\left\|\gamma(s,z)\right\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}s,
\tag{2.35}
$$

for any $t>0$ and

$$
\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\left\|\mathbf{u}^R(t)\right\|_{\mathrm{L}^2}^2\Big]\leqslant C\left(\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\right],T,K_1,K_3\right),
\tag{2.36}
$$

for any $T>0$.

Proof. Let us define the sequence of stopping times τ_M to be

$$
\tau_M:=\inf_{t\geqslant 0}\left\{t:\left\|\mathbf{u}^R(t)\right\|_{\mathrm{L}^2}\geqslant M\right\}.
\tag{2.37}
$$

Let us apply the Itô's formula (Theorem 3.7.2, [30], Theorem 4.4, [41], Section 2.3, [32]) to $\|\mathbf{u}^R(\cdot)\|_{\mathrm{L}^2}^2$ to obtain

$$
\begin{aligned}
\left\|\mathbf{u}^R(t\wedge\tau_M)\right\|_{\mathrm{L}^2}^2=\left\|\mathbf{u}^R(0)\right\|_{\mathrm{L}^2}^2&-2\int_0^{t\wedge\tau_M}\left(\mathcal{S}_R\mathrm{P}_{\mathcal{H}}\left(\mathbf{u}^R(s)\cdot\nabla\right)\mathbf{u}^R(s),\mathbf{u}^R(s)\right)_{\mathrm{L}^2}\mathrm{d}s\\
&+2\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\left(\mathcal{S}_R\Phi e_j,\mathbf{u}^R(s)\right)_{\mathrm{L}^2}\mathrm{d}\beta_j(s)\\
&+\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\|\mathcal{S}_R\Phi e_j\|_{\mathrm{L}^2}^2\,\mathrm{d}s+\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\left\|\mathcal{S}_R\gamma(s,z)\right\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}s\\
&+\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\left[2\left(\mathcal{S}_R\gamma(s-,z),\mathbf{u}^R(s-)\right)_{\mathrm{L}^2}+\left\|\mathcal{S}_R\gamma(s,z)\right\|_{\mathrm{L}^2}^2\right]\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z),
\end{aligned}
\tag{2.38}
$$

for $0 \leqslant t \leqslant T$. By using the divergence free condition (see (2.22)) and $\mathcal{S}_R\mathbf{u}^R = \mathbf{u}^R$ in $\mathcal{H}_R$, we get

$$\begin{aligned}\big(\mathcal{S}_R \mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R, \mathbf{u}^R\big)_{\mathrm{L}^2} &= \big(\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R, \mathcal{S}_R\mathbf{u}^R\big)_{\mathrm{L}^2} = \big(\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R, \mathbf{u}^R\big)_{\mathrm{L}^2}\\ &= \big(\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R, \mathrm{P}_{\mathcal{H}}\mathbf{u}^R\big)_{\mathrm{L}^2} = \big(\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R, \mathbf{u}^R\big)_{\mathrm{L}^2} = 0. \end{aligned}\tag{2.39}$$

By using the cut off property (2.16), we obtain

$$\begin{aligned}&\sum_{j=1}^{\infty}\|\mathcal{S}_R\Phi e_j\|_{\mathrm{L}^2}^2 + \int_{\mathrm{Z}}\big\|\mathcal{S}_R\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\\ &\quad\leqslant \sum_{j=1}^{\infty}\|\Phi e_j\|_{\mathrm{L}^2}^2 + \int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z) = \mathrm{Tr}\big(\Phi^*\Phi\big) + \int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z). \end{aligned}\tag{2.40}$$

On Substituting (2.39) and (2.40) in (2.38), taking expectation, and noting that the stochastic integrals

$$\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\big(\mathcal{S}_R\Phi e_j, \mathbf{u}^R(s)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(s) \quad \text{and}$$

$$\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big[\big(\mathcal{S}_R\gamma(s-,z), \mathbf{u}^R(s-)\big)_{\mathrm{L}^2} + \big\|\mathcal{S}_R\gamma(s,z)\big\|_{\mathrm{L}^2}^2\big]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)$$

are local martingales having zero expectation, we get

$$\begin{aligned}\mathbb{E}\big[\big\|\mathbf{u}^R(t\wedge\tau_M)\big\|_{\mathrm{L}^2}^2\big] &\leqslant \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big] + \mathbb{E}\bigg[\int_0^{t\wedge\tau_M}\mathrm{Tr}\big(\Phi^*\Phi\big)\,\mathrm{d}s\bigg] + \mathbb{E}\bigg[\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big\|\gamma(s,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}s\bigg]\\ &\leqslant \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big] + t\,\mathrm{Tr}\big(\Phi^*\Phi\big) + \int_0^t\int_{\mathrm{Z}}\big\|\gamma(s,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}s, \end{aligned}\tag{2.41}$$

where we used the fact that $\|\mathbf{u}^R(0)\|_{\mathrm{L}^2}^2 = \|\mathcal{S}_R\mathbf{u}_0\|_{\mathrm{L}^2}^2 \leqslant \|\mathbf{u}_0\|_{\mathrm{L}^2}^2$. Note that the right hand side of the inequality (2.41) is independent of M. On passing $M\to\infty$, $t\wedge\tau_M\to t$ and then using the dominated convergence theorem in (2.41), we have (2.35).

Let us take supremum from 0 to T in (2.38) and then take expectation to get

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{L}^2}^2\Big]\\ &\quad\leqslant \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big] + \mathbb{E}\bigg[\int_0^{T\wedge\tau_M}\mathrm{Tr}\big(\Phi^*\Phi\big)\,\mathrm{d}t + \int_0^{T\wedge\tau_M}\int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}t,\mathrm{d}z)\bigg]\\ &\qquad + 2\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\sum_{j=1}^{\infty}\big(\mathcal{S}_R\Phi e_j, \mathbf{u}^R(s)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(s)\bigg|\bigg]\\ &\qquad + 2\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\int_{\mathrm{Z}}\big(\mathcal{S}_R\gamma(s-,z), \mathbf{u}^R(s-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\bigg|\bigg]\\ &\quad\leqslant \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big] + \mathrm{Tr}\big(\Phi^*\Phi\big)T + \int_0^T\int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t + (I_1) + (I_2), \end{aligned}\tag{2.42}$$

where

$$(I_1) = 2\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\sum_{j=1}^{\infty}\big(\mathcal{S}_R\Phi e_j, \mathbf{u}^R(s)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(s)\bigg|\bigg], \quad \text{and} \tag{2.43}$$

$$(I_2) = 2\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\int_{\mathrm{Z}}\big(\mathcal{S}_R\gamma(s-,z), \mathbf{u}^R(s-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\bigg|\bigg]. \tag{2.44}$$

In the last step of (2.42), we used

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^T\mathrm{Tr}\big(\Phi^*\Phi\big)\,\mathrm{d}t + \int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}t,\mathrm{d}z)\bigg]\\ &\quad = \int_0^T\mathrm{Tr}\big(\Phi^*\Phi\big)\,\mathrm{d}t + \int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t,\end{aligned} \tag{2.45}$$

since $\int_0^T\mathrm{Tr}(\Phi^*\Phi)\,\mathrm{d}t + \int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}t,\mathrm{d}z)$ is the Meyer process of $\mathbf{u}^R(\cdot)$ and $\int_0^T\mathrm{Tr}(\Phi^*\Phi)\,\mathrm{d}t + \int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t$ is the quadratic variation process of $\mathbf{u}^R(\cdot)$ (Section 2.3, [32]).

Let us apply the Burkholder–Davis–Gundy inequality (Theorem 1.1, [31]), Hölder inequality and Young's inequality to the term (I_1) in (2.42) to get

$$\begin{aligned}(I_1) &\leqslant 2\sqrt{2}\sum_{j=1}^{\infty}\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\|\mathcal{S}_R\Phi e_j\|_{\mathrm{L}^2}^2\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\\ &\leqslant 2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant T\wedge\tau_M}\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\Big)^{\frac{1}{2}}\sum_{j=1}^{\infty}\bigg(\int_0^{T\wedge\tau_M}\|\Phi e_j\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]\\ &\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\Big) + 8\int_0^T\mathrm{Tr}\big(\Phi^*\Phi\big)\,\mathrm{d}t.\end{aligned} \tag{2.46}$$

Let us now apply the Burkholder–Davis–Gundy inequality, Hölder inequality and Young's inequality to the term (I_2) in (2.42) to obtain

$$\begin{aligned}(I_2) &\leqslant 2\sqrt{2}\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\int_{\mathrm{Z}}\|\mathcal{S}_R\gamma(t,z)\|_{\mathrm{L}^2}^2\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg)^{\frac{1}{2}}\\ &\leqslant 2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\Big)^{\frac{1}{2}}\bigg(\int_0^{T\wedge\tau_M}\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]\\ &\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\Big) + 8\int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t.\end{aligned} \tag{2.47}$$

Substituting (2.47) and (2.46) in (2.42), we get

$$\frac{1}{2}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\|\mathbf{u}^R(t)\|_{\mathrm{L}^2}^2\Big] \leqslant \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big] + 9\,\mathrm{Tr}\big(\Phi^*\Phi\big)T + 9\int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t. \tag{2.48}$$

Hence, we have

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{L}^2}^2\Big]\leqslant 2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big]+18(K_1T+K_3). \tag{2.49}$$

The right hand side of the inequality (2.48) is independent of M, on passing $M\to\infty$, $T\wedge\tau_M\to T$ and applying the dominated convergence theorem, we get (2.36). □

Remark 2.10. Proposition 2.9 is also true for the truncated stochastic Euler equation with the cut off function $\psi_N(\cdot)$ (see (2.31)–(2.32)) as $(\psi_N(k^R)\mathcal{S}_R\mathrm{P}_{\mathcal{H}}(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R,\mathbf{u}^R)_{\mathrm{L}^2}=0$. Note that the L^2-energy estimate for the truncated system (2.25)–(2.26) exists for all $T>0$.

Proposition 2.11 (H^s-energy estimate). *Let the given initial data $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathrm{H}^s(\mathbb{R}^n))$, for $s>n/2+1$ be $\mathscr{F}_0$-measurable. Then for any $t\in(0,T)$, we have*

$$\mathbb{E}\big[\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\big]\leqslant\left(\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{L}^2}^2\big]+t\,\mathrm{Tr}\big((\mathrm{J}^s\Phi)^*\mathrm{J}^s\Phi\big)+\int_0^t\int_{\mathrm{Z}}\|\gamma(s,z)\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}s\right)e^{2C_Nt}, \tag{2.50}$$

and for any $T>0$, we have

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big]\leqslant C\big(\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big],K_2,K_4,N,T\big) \tag{2.51}$$

and the quantity on the right hand side of the inequality (2.51) is independent of R.

Proof. An application of the operator J^s on the truncated Eq. (2.31) gives

$$\mathrm{d}\mathrm{J}^s\mathbf{u}^R=-\psi_N\big(k^R\big)\mathcal{S}_R\mathrm{J}^s\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\,\mathrm{d}t+\sum_{j=1}^{\infty}\mathcal{S}_R\mathrm{J}^s\Phi e_j\,\mathrm{d}\beta_j(t)+\int_{\mathrm{Z}}\mathcal{S}_R\mathrm{J}^s\gamma(t-,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z). \tag{2.52}$$

Let us define the sequence of stopping times τ_M to be

$$\tau_M=\inf_{t\geqslant 0}\big\{t:\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}\geqslant M\big\}. \tag{2.53}$$

Let us apply the Itô's formula to $\|\mathrm{J}^s\mathbf{u}^R(\cdot)\|_{\mathrm{L}^2}^2$ to obtain

$$\begin{aligned}\big\|\mathrm{J}^s\mathbf{u}^R(t\wedge\tau_M)\big\|_{\mathrm{L}^2}^2&=\big\|\mathrm{J}^s\mathbf{u}^R(0)\big\|_{\mathrm{L}^2}^2-2\int_0^{t\wedge\tau_M}\big(\psi_N\big(k^R\big)\mathcal{S}_R\mathrm{J}^s\mathrm{P}_{\mathcal{H}}\big[\big(\mathbf{u}^R(r)\cdot\nabla\big)\mathbf{u}^R(r)\big],\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\mathrm{d}r\\&\quad+2\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\big(\mathcal{S}_R\mathrm{J}^s\Phi e_j,\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\mathrm{d}\beta_j(r)+\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\big\|\mathcal{S}_R\mathrm{J}^s\Phi e_j\big\|_{\mathrm{L}^2}^2\mathrm{d}r\\&\quad+2\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big(\mathcal{S}_R\mathrm{J}^s\gamma(r-,z),\mathrm{J}^s\mathbf{u}^R(r-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\\&\quad+\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big\|\mathcal{S}_R\mathrm{J}^s\gamma(r,z)\big\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z),\end{aligned} \tag{2.54}$$

for $0 \leqslant t \leqslant T$. Let us consider the term $\sum_{j=1}^{\infty}\|\mathcal{S}_R\mathrm{J}^s\Phi e_j\|_{\mathrm{L}^2}^2 + \int_{\mathrm{Z}}\|\mathcal{S}_R\mathrm{J}^s\gamma(r,z)\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z)$ and use the cut off property (2.16) to get

$$\begin{aligned}
&\sum_{j=1}^{\infty}\big\|\mathcal{S}_R\mathrm{J}^s\Phi e_j\big\|_{\mathrm{L}^2}^2 + \int_{\mathrm{Z}}\big\|\mathcal{S}_R\mathrm{J}^s\gamma(r,z)\big\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z)\\
&\quad = \mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big) + \int_{\mathrm{Z}}\big\|\gamma(r,z)\big\|_{\mathrm{H}^s}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z).
\end{aligned} \tag{2.55}$$

Now let us take the nonlinear term $(\psi_N(k^R)\mathcal{S}_R\mathrm{J}^s\mathrm{P}_{\mathcal{H}}[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R],\mathrm{J}^s\mathbf{u}^R)_{\mathrm{L}^2}$ and use the Kato–Ponce commutator estimates (Corollary 2.4) to obtain

$$\begin{aligned}
&\big|\big(\psi_N\big(k^R\big)\mathcal{S}_R\mathrm{J}^s\mathrm{P}_{\mathcal{H}}\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big],\mathrm{J}^s\mathbf{u}^R\big)_{\mathrm{L}^2}\big|\\
&\quad = \big|\big(\psi_N\big(k^R\big)\mathrm{J}^s\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big],\mathrm{J}^s\mathrm{P}_{\mathcal{H}}\mathbf{u}^R\big)_{\mathrm{L}^2}\big|\\
&\quad \leqslant \big\|\psi_N\big(k^R\big)\mathrm{J}^s\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big]\big\|_{\mathrm{L}^2}\big\|\mathrm{J}^s\mathbf{u}^R\big\|_{\mathrm{L}^2}\\
&\quad \leqslant \big|\psi_N\big(k^R\big)\big|\big\|\nabla\mathbf{u}^R\big\|_{\mathrm{L}^\infty}\big\|\mathbf{u}^R\big\|_{\mathrm{H}^s}^2 \leqslant C_N\big\|\mathbf{u}^R\big\|_{\mathrm{H}^s}^2,
\end{aligned} \tag{2.56}$$

where we used $\mathcal{S}_R\mathbf{u}^R = \mathbf{u}^R$ in $\mathcal{H}_R$ and (2.4). Let us use (2.55) and (2.56) in (2.54) to get

$$\begin{aligned}
\big\|\mathbf{u}^R(t\wedge\tau_M)\big\|_{\mathrm{H}^s}^2 \leqslant{}& \big\|\mathbf{u}^R(0)\big\|_{\mathrm{H}^s}^2 + 2C_N\int_0^{t\wedge\tau_M}\big\|\mathbf{u}^R(r)\big\|_{\mathrm{H}^s}^2\,\mathrm{d}r\\
&+ 2\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\big(\mathcal{S}_R\mathrm{J}^s\Phi e_j,\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)\\
&+ \int_0^{t\wedge\tau_M}\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big)\,\mathrm{d}r + \int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big\|\gamma(r,z)\big\|_{\mathrm{H}^s}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z)\\
&+ 2\int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big(\mathcal{S}_R\mathrm{J}^s\gamma(r-,z),\mathrm{J}^s\mathbf{u}^R(r-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z).
\end{aligned} \tag{2.57}$$

Let us take the expectation in (2.57) to find

$$\begin{aligned}
\mathbb{E}\big[\big\|\mathbf{u}^R(t\wedge\tau_M)\big\|_{\mathrm{H}^s}^2\big] \leqslant{}& \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big] + 2C_N\mathbb{E}\bigg(\int_0^{t\wedge\tau_M}\big\|\mathbf{u}^R(r)\big\|_{\mathrm{H}^s}^2\,\mathrm{d}r\bigg)\\
&+ t\,\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big) + \int_0^t\int_{\mathrm{Z}}\big\|\gamma(r,z)\big\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}r,
\end{aligned} \tag{2.58}$$

where we used (2.45), $\|\mathbf{u}^R(0)\|_{\mathrm{H}^s}^2 = \|\mathcal{S}_R\mathbf{u}_0\|_{\mathrm{H}^s}^2 \leqslant \|\mathbf{u}_0\|_{\mathrm{H}^s}^2$, and the fact that

$$\int_0^{t\wedge\tau_M}\sum_{j=1}^{\infty}\big(\mathcal{S}_R\mathrm{J}^s\Phi e_j,\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r) \quad\text{and}\quad \int_0^{t\wedge\tau_M}\int_{\mathrm{Z}}\big(\mathcal{S}_R\mathrm{J}^s\gamma(r-,z),\mathrm{J}^s\mathbf{u}^R(r-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)$$

are local martingales having zero expectation. Let us apply the Gronwall's inequality in (2.58) to get

$$
\begin{aligned}
&\mathbb{E}\big[\big\|\mathbf{u}^R(t\wedge\tau_M)\big\|_{\mathrm{H}^s}^2\big]\\
&\quad\leqslant\bigg(\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+t\,\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big)+\int_0^t\int_{\mathrm{Z}}\big\|\gamma(s,z)\big\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}s\bigg)e^{2C_Nt},
\end{aligned}
\tag{2.59}
$$

for any $t\in(0,T)$. On passing $M\to\infty$, $t\wedge\tau_M\to t$ and using the dominated convergence theorem in (2.59), we get (2.50).

Let us take the supremum from 0 to T in (2.57) and then take the expectation to obtain

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)\\
&\quad\leqslant\mathbb{E}\big(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big)+2C_N\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\,\mathrm{d}t\bigg)\\
&\qquad+\int_0^T\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big)\,\mathrm{d}t+\int_0^T\int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\\
&\qquad+2\mathbb{E}\bigg(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\sum_{j=1}^{\infty}\int_0^t\big(\mathcal{S}_R\mathrm{J}^s\Phi e_j,\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)\bigg|\bigg)\\
&\qquad+2\mathbb{E}\bigg(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\int_{\mathrm{Z}}\big(\mathcal{S}_R\mathrm{J}^s\gamma(r-,z),\mathrm{J}^s\mathbf{u}^R(r-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\bigg|\bigg).
\end{aligned}
\tag{2.60}
$$

Let us take

$$
(I_3)=2\mathbb{E}\bigg(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\sum_{j=1}^{\infty}\int_0^t\big(\mathcal{S}_R\mathrm{J}^s\Phi e_j,\mathrm{J}^s\mathbf{u}^R(r)\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)\bigg|\bigg),
$$

and

$$
(I_4)=2\mathbb{E}\bigg(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\bigg|\int_0^t\int_{\mathrm{Z}}\big(\mathcal{S}_R\mathrm{J}^s\gamma(r-,z),\mathrm{J}^s\mathbf{u}^R(r-)\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\bigg|\bigg).
$$

Now by applying the Burkholder–Davis–Gundy inequality, Hölder inequality and Young's inequality to the term (I_3), we get

$$
\begin{aligned}
(I_3)&\leqslant2\sqrt{2}\sum_{j=1}^{\infty}\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\big\|\mathcal{S}_R\mathrm{J}^s\Phi e_j\big\|_{\mathrm{L}^2}^2\big\|\mathrm{J}^s\mathbf{u}^R(t)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\\
&\leqslant2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)^{\frac{1}{2}}\sum_{j=1}^{\infty}\bigg(\int_0^{T\wedge\tau_M}\big\|\mathrm{J}^s\Phi e_j\big\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]\\
&\leqslant\frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)+8\int_0^T\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big)\,\mathrm{d}t.
\end{aligned}
\tag{2.61}
$$

Once again by applying the Burkholder–Davis–Gundy inequality, Hölder inequality and Young's inequality to the term (I_4), we get

$$\begin{aligned}(I_4) &\leqslant 2\sqrt{2}\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\int_{\mathrm{Z}}\big\|\mathcal{S}_R\mathrm{J}^s\gamma(t,z)\big\|_{\mathrm{L}^2}^2\big\|\mathrm{J}^s\mathbf{u}^R(t)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg)^{\frac{1}{2}}\\ &\leqslant 2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)^{\frac{1}{2}}\bigg(\int_0^{T\wedge\tau_M}\int_{\mathrm{Z}}\big\|\mathrm{J}^s\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]\\ &\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)+8\int_0^T\int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}t.\end{aligned}\tag{2.62}$$

By using (2.61) and (2.62) in (2.60), we obtain

$$\begin{aligned}\frac{1}{2}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big) &\leqslant \mathbb{E}\big(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big)+2C_N\mathbb{E}\bigg(\int_0^{T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\,\mathrm{d}t\bigg)\\ &\quad+9\bigg[\int_0^T\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\mathrm{J}^s\Phi\big)\,\mathrm{d}t+\int_0^T\int_{\mathrm{Z}}\big\|\gamma(t,z)\big\|_{\mathrm{H}^s}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg].\end{aligned}\tag{2.63}$$

Thus from (2.63), we have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)\\ &\quad\leqslant 2\mathbb{E}\big(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big)+18(K_2T+K_4)+4C_N\int_0^T\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t\wedge\tau_M}\big\|\mathbf{u}^R(r)\big\|_{\mathrm{H}^s}^2\Big)\,\mathrm{d}t.\end{aligned}\tag{2.64}$$

An application of the Gronwall's inequality yields

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_M}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)\leqslant\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18(K_2T+K_4)\big)e^{4C_NT}.\tag{2.65}$$

Note that the right hand side of the inequality (2.65) is independent of M and R, on passing $M\to\infty$, we see that $T\wedge\tau_M\to T$. Hence by using the dominated convergence theorem, from (2.65), we finally obtain

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)\leqslant\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18(K_2T+K_4)\big)e^{4C_NT},\tag{2.66}$$

where C_N is a constant depending on N and independent of R. □

Remark 2.12. From Proposition 2.11, it is clear that for the stopping time

$$\tau_N:=\inf_{t\geqslant 0}\big\{t:\big\|\nabla\mathbf{u}^R(t)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^{s-1}}\geqslant N\big\},\tag{2.67}$$

the quantity $\mathbb{E}[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}^R(t)\|_{\mathrm{H}^s}^2]$ is uniformly bounded and the bound is independent of R. Also the solution of the problem (2.25)–(2.26) can be defined up to a time $T\wedge\tau_N$, where τ_N is defined in (2.67) and it can be easily seen that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^s}^2\Big)\leqslant\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18(K_2T+K_4)\big)e^{4NT}. \tag{2.68}$$

In the estimate (2.68), we cannot take $N\to\infty$, as the right hand side of the inequality (2.68) is exponentially growing in N.

Theorem 2.13. *Let $0<\delta<1$ be given. Then, we have*

$$\mathbb{P}\{\tau_N>\delta\}\geqslant 1-C\delta^2\big\{2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18\big(K_2\delta+K_4(\delta)\big)\big\}, \tag{2.69}$$

for some positive constant C independent of δ.

Proof. For the given $0<\delta<1$, there exists a positive integer N such that

$$\frac{1}{N+1}\leqslant\delta<\frac{1}{N}.$$

Thus, we can choose a δ so that $(\mathbf{u}^R, T\wedge\tau_N)$ is a local strong solution of (2.25)–(2.26) with

$$\tau_N:=\inf_{t\geqslant 0}\big\{t:\big\|\nabla\mathbf{u}^R(t)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{u}^R(t)\big\|_{\mathrm{H}^{s-1}}\geqslant N\big\}.$$

Then it can be easily seen that

$$\begin{aligned}\mathbb{P}\{\tau_N>\delta\}&\geqslant\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant\delta}\big(\big\|\nabla\mathbf{u}^R(t\wedge\tau_N)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{\mathrm{H}^{s-1}}\big)<N\Big\}\\&\geqslant\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant\delta}\big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{\mathrm{H}^s}<\mathcal{K}N\Big\},\end{aligned} \tag{2.70}$$

where $\mathcal{K}$ is a positive constant defined by

$$\mathcal{K}=\sup\big\{C\in\mathbb{R}^+\big|C\big(\|\nabla\mathbf{v}\|_{\mathrm{L}^\infty}+\|\mathbf{v}\|_{\mathrm{H}^{s-1}}\big)\leqslant\|\mathbf{v}\|_{\mathrm{H}^s},\forall\,\mathbf{v}\in\mathrm{H}^s\big(\mathbb{R}^n\big)\text{ with }\nabla\cdot\mathbf{v}=0\big\}. \tag{2.71}$$

By using (2.68), we get

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\delta}\big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{\mathrm{H}^s}^2\Big)\leqslant\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]+18\big(K_2\delta+K_4(\delta)\big)\big)e^{4N\delta}, \tag{2.72}$$

where $K_4(\delta) = \int_0^\delta \int_Z \|\gamma(t,z)\|_{H^s}^2 \lambda(dz)\,dt$. Then by making use of Markov's inequality and (2.72), we obtain

$$\begin{aligned}
\mathbb{P}\{\tau_N > \delta\} &\geqslant \mathbb{P}\Big\{\sup_{0\leqslant t\leqslant\delta} \big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{H^s} < \mathcal{K}N\Big\} \\
&= \mathbb{P}\Big\{\sup_{0\leqslant t\leqslant\delta} \big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{H^s}^2 < \mathcal{K}^2N^2\Big\} \\
&\geqslant 1 - \frac{1}{\mathcal{K}^2N^2}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\delta} \big\|\mathbf{u}^R(t\wedge\tau_N)\big\|_{H^s}^2\Big) \\
&\geqslant 1 - \frac{1}{\mathcal{K}^2N^2}\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{H^s}^2\big] + 18\big(K_2\delta + K_4(\delta)\big)\big)e^{4N\delta} \\
&\geqslant 1 - C\delta^2\big\{2\mathbb{E}\big[\|\mathbf{u}_0\|_{H^s}^2\big] + 18\big(K_2\delta + K_4(\delta)\big)\big\},
\end{aligned} \tag{2.73}$$

where C is a constant independent of $\mathbf{u}$ and δ. □

Similar ideas for proving the positivity of stopping time for stochastic quasilinear hyperbolic system can be found in Theorem 1.3 [26].

3. Existence and uniqueness

We are now ready to prove the existence of local strong solutions of the stochastic Euler equations with Lévy noise (see (2.7)–(2.8)). In order to establish this we first prove that the solutions $(\mathbf{u}^R, T\wedge\tau_N)$ of smoothed version of the Eqs (2.7)–(2.8) is a Cauchy sequence in the L^2-norm as $R\to\infty$ with probability 1. Let $(\mathbf{u}^R, T\wedge\tau_N)$ be a local strong solution of the truncated Eqs (2.25)–(2.26), where $\tau_N(\omega)$ is the stopping time defined in (2.67).

Proposition 3.1. *Let $\mathbf{u}_0 \in L^2(\Omega; H^s(\mathbb{R}^n))$ be $\mathscr{F}_0$-measurable for $s > n/2+1$ with $\mathbb{E}[\|\mathbf{u}_0\|_{H^s}^2] < \infty$. Then, the family of local strong solutions $(\mathbf{u}^R, T\wedge\tau_N)$ of (2.25)–(2.26) is Cauchy (as $R\to\infty$) in $L^2(\Omega; L^\infty(0, T\wedge\tau_N; L^2(\mathbb{R}^n)))$, i.e.,*

$$\lim_{R\to\infty}\sup_{R'\geqslant R}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R - \mathbf{u}^{R'}\big\|_{L^2}^2\Big) = 0, \tag{3.1}$$

where τ_N is the stopping time defined in (2.67).

Proof. Let $(\mathbf{u}^R, T\wedge\tau_{N_1})$ and $(\mathbf{u}^{R'}, T\wedge\tau_{N_2})$ be two local strong solutions of (2.25)–(2.26) in $\mathcal{H}_R$ and $\mathcal{H}_{R'}$ respectively. Let us define $\tau_N := \tau_{N_1}\wedge\tau_{N_2}$ and take the difference between the equations for the processes $\mathbf{u}^R(\cdot)$ and $\mathbf{u}^{R'}(\cdot)$ $(R' > R)$ to get

$$\begin{aligned}
d\big(\mathbf{u}^R - \mathbf{u}^{R'}\big) &= -\big(\mathcal{S}_R \mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R - \mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^{R'}\cdot\nabla\big)\mathbf{u}^{R'}\big)\,dt \\
&\quad + \sum_{j=1}^\infty (\mathcal{S}_R - \mathcal{S}_{R'})\Phi e_j\, d\beta_j(t) + \int_Z (\mathcal{S}_R - \mathcal{S}_{R'})\gamma(t-,z)\widetilde{\mathcal{N}}(dt,dz).
\end{aligned} \tag{3.2}$$

Let us apply Itô's formula to $\|(\mathbf{u}^R-\mathbf{u}^{R'})(\cdot)\|^2_{\mathrm{L}^2}$ to obtain

$$\begin{aligned}
&\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t\wedge\tau_N)\big\|^2_{\mathrm{L}^2}\\
&\quad=\big\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\big\|^2_{\mathrm{L}^2}-2\int_0^{t\wedge\tau_N}\big(\big[\mathcal{S}_R\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R-\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^{R'}\cdot\nabla\big)\mathbf{u}^{R'}\big],\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\,\mathrm{d}r\\
&\qquad+2\int_0^{t\wedge\tau_N}\sum_{j=1}^{\infty}\big((\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)+\int_0^{t\wedge\tau_N}\sum_{j=1}^{\infty}\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\big\|^2_{\mathrm{L}^2}\,\mathrm{d}r\\
&\qquad+2\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big((\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r-,z),\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\\
&\qquad+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r,z)\big\|^2_{\mathrm{L}^2}\mathcal{N}(\mathrm{d}r,\mathrm{d}z),
\end{aligned}\tag{3.3}$$

for $0\leqslant t\leqslant T$. We write the term $([\mathcal{S}_R\mathrm{P}_{\mathcal{H}}(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R-\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}(\mathbf{u}^{R'}\cdot\nabla)\mathbf{u}^{R'}],\mathbf{u}^R-\mathbf{u}^{R'})_{\mathrm{L}^2}$ from (3.3) as

$$\begin{aligned}
&\big(\big[\mathcal{S}_R\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R-\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^{R'}\cdot\nabla\big)\mathbf{u}^{R'}\big],\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\\
&\quad=\big((\mathcal{S}_R-\mathcal{S}_{R'})\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\\
&\qquad+\big(\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)\cdot\nabla\big)\mathbf{u}^R,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\\
&\qquad+\big(\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}^{R'}\cdot\nabla\big)\big(\mathbf{u}^R-\mathbf{u}^{R'}\big),\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}.
\end{aligned}\tag{3.4}$$

The third term from the right hand side of the equality (3.4) is zero, by using the divergence free condition (see (2.12) and (2.22)). For $R'>R$ and for the stopping time defined in (2.67), let us take the first term from the right hand side of the equality (3.4) and use the Hölder's inequality, cut off property (see (2.18) and (2.30), with $k=\varepsilon$, $0<\varepsilon<s-2$) and the algebra property of H^{s-1}-norm to obtain

$$\begin{aligned}
&\big|\big((\mathcal{S}_R-\mathcal{S}_{R'})\mathrm{P}_{\mathcal{H}}\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big],\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\big|\\
&\quad\leqslant\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{L}^2}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\\
&\quad\leqslant\frac{C}{R^\varepsilon}\big\|\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{H}^\varepsilon}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\leqslant\frac{C}{R^\varepsilon}\big\|\nabla\cdot\big(\mathbf{u}^R\otimes\mathbf{u}^R\big)\big\|_{\mathrm{H}^\varepsilon}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\\
&\quad\leqslant\frac{C}{R^\varepsilon}\big\|\nabla\cdot\big(\mathbf{u}^R\otimes\mathbf{u}^R\big)\big\|_{\mathrm{H}^{s-2}}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\leqslant\frac{C}{R^\varepsilon}\big\|\big(\mathbf{u}^R\otimes\mathbf{u}^R\big)\big\|_{\mathrm{H}^{s-1}}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\\
&\quad\leqslant\frac{C}{R^\varepsilon}\big\|\mathbf{u}^R\big\|^2_{\mathrm{H}^{s-1}}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\leqslant\frac{CN_1^2}{R^\varepsilon}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}.
\end{aligned}\tag{3.5}$$

Note that $\nabla\cdot(\mathbf{u}\otimes\mathbf{u})=(\mathbf{u}\cdot\nabla)\mathbf{u}$ for $\nabla\cdot\mathbf{u}=0$. By using the Cauchy–Schwarz inequality and Hölder's inequality, we estimate the second term from the right hand side of the equality (3.4) as

$$\begin{aligned}
&\big|\big(\mathcal{S}_{R'}\mathrm{P}_{\mathcal{H}}\big(\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)\cdot\nabla\big)\mathbf{u}^R,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\big|\\
&\quad\leqslant\big\|\big(\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{L}^2}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\\
&\quad\leqslant C\big\|\nabla\mathbf{u}^R\big\|_{\mathrm{L}^\infty}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}^2\leqslant CN_1\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}^2.
\end{aligned}\tag{3.6}$$

Let us combine (3.5), (3.6) and use it in (3.3) to obtain

$$\begin{aligned}
&\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t\wedge\tau_N)\big\|_{\mathrm{L}^2}^2\\
&\quad\leqslant\big\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\big\|_{\mathrm{L}^2}^2+\frac{2CN_1^2}{R^\varepsilon}\int_0^{t\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}\,\mathrm{d}r+2CN_1\int_0^{t\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}^{R'}\big\|_{\mathrm{L}^2}^2\,\mathrm{d}r\\
&\qquad+2\int_0^{t\wedge\tau_N}\sum_{j=1}^\infty\big((\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)+\int_0^{t\wedge\tau_N}\sum_{j=1}^\infty\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\big\|_{\mathrm{L}^2}^2\,\mathrm{d}r\\
&\qquad+2\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big((\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r-,z),\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\\
&\qquad+\int_0^{t\wedge\tau_N}\int_{\mathrm{Z}}\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r,z)\big\|_{\mathrm{L}^2}^2\mathcal{N}(\mathrm{d}r,\mathrm{d}z).
\end{aligned}\tag{3.7}$$

Let us take the supremum from 0 to T in (3.7) and then take the expectation to get

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\big\|_{\mathrm{L}^2}^2\Big)\\
&\quad\leqslant\mathbb{E}\big(\big\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\big\|_{\mathrm{L}^2}^2\big)\\
&\qquad+\frac{2CN_1^2}{R^\varepsilon}\mathbb{E}\bigg(\int_0^{\widetilde{\tau}_N}\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\big\|_{\mathrm{L}^2}\,\mathrm{d}t\bigg)+2CN_1\mathbb{E}\bigg(\int_0^{\widetilde{\tau}_N}\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg)\\
&\qquad+\sum_{j=1}^\infty\int_0^T\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\big\|_{\mathrm{L}^2}^2\,\mathrm{d}t+\int_0^T\int_{\mathrm{Z}}\big\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\big\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\\
&\qquad+2\mathbb{E}\Bigg[\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\bigg|\sum_{j=1}^\infty\int_0^t\big((\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)\bigg|\Bigg]\\
&\qquad+2\mathbb{E}\Bigg[\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\bigg|\int_0^t\int_{\mathrm{Z}}\big((\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r-,z),\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\bigg|\Bigg],
\end{aligned}\tag{3.8}$$

where $\widetilde{\tau}_N:=T\wedge\tau_{N_1}\wedge\tau_{N_2}$. Let us denote

$$(I_5)=2\mathbb{E}\Bigg[\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\bigg|\sum_{j=1}^\infty\int_0^t\big((\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j,\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\,\mathrm{d}\beta_j(r)\bigg|\Bigg],$$

and

$$(I_6) = 2\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\bigg|\int_0^t\int_Z\big((\mathcal{S}_R-\mathcal{S}_{R'})\gamma(r-,z),\mathbf{u}^R-\mathbf{u}^{R'}\big)_{\mathrm{L}^2}\widetilde{\mathcal{N}}(\mathrm{d}r,\mathrm{d}z)\bigg|\bigg].$$

Now we apply the Burkholder–Davis–Gundy inequality, Hölder inequality and Young's inequality to the term (I_5) to obtain

$$\begin{aligned}
(I_5) &\leqslant 2\sqrt{2}\sum_{j=1}^{\infty}\mathbb{E}\bigg[\int_0^{\widetilde{\tau}_N}\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\|_{\mathrm{L}^2}^2\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\,\mathrm{d}t\bigg]^{\frac{1}{2}}\\
&\leqslant 2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\Big)^{\frac{1}{2}}\sum_{j=1}^{\infty}\Big(\int_0^{\widetilde{\tau}_N}\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\|_{\mathrm{L}^2}^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\bigg]\\
&\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\Big)+8\sum_{j=1}^{\infty}\int_0^T\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\|_{\mathrm{L}^2}^2\,\mathrm{d}t.
\end{aligned}\tag{3.9}$$

By applying the Burkholder–Davis–Gundy inequality, Hölder inequality and Young's inequality to the term (I_6), we get

$$\begin{aligned}
(I_6) &\leqslant 2\sqrt{2}\mathbb{E}\bigg[\int_0^{\widetilde{\tau}_N}\int_Z\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\|_{\mathrm{L}^2}^2\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg]^{\frac{1}{2}}\\
&\leqslant 2\sqrt{2}\mathbb{E}\bigg[\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\Big)^{\frac{1}{2}}\Big(\int_0^{\widetilde{\tau}_N}\int_Z\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t\Big)^{\frac{1}{2}}\bigg]\\
&\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{\mathrm{L}^2}^2\Big)+8\int_0^T\int_Z\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t.
\end{aligned}\tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8) and then using the inequality $2ab\leqslant a^2+b^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\big\|_{\mathrm{L}^2}^2\Big)\\
&\quad\leqslant \mathbb{E}\big(\big\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\big\|_{\mathrm{L}^2}^2\big)+\frac{CN_1^2T}{R^{\varepsilon}}+\Big(\frac{CN_1^2}{R^{\varepsilon}}+2CN_1\Big)\mathbb{E}\Big(\int_0^{\widetilde{\tau}_N}\big\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}t\Big)\\
&\qquad+9\sum_{j=1}^{\infty}\int_0^T\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\|_{\mathrm{L}^2}^2\,\mathrm{d}t+9\int_0^T\int_Z\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t.
\end{aligned}\tag{3.11}$$

Let us take the term $\mathbb{E}(\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\|_{\mathrm{L}^2}^2)$ from the inequality (3.11) and use (2.18) to find

$$\mathbb{E}\big(\big\|\mathbf{u}^R(0)-\mathbf{u}^{R'}(0)\big\|_{\mathrm{L}^2}^2\big)=\mathbb{E}\big(\big\|\mathcal{S}_R\mathbf{u}_0-\mathcal{S}_{R'}\mathbf{u}_0\big\|_{\mathrm{L}^2}^2\big)\leqslant\frac{C}{R^{\varepsilon}}\mathbb{E}\big(\|\mathbf{u}_0\|_{\mathrm{H}^{\varepsilon}}^2\big)\leqslant\frac{C}{R^{\varepsilon}}\mathbb{E}\big(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big).\tag{3.12}$$

Let us take final term from the right hand side of the inequality (3.11) and simplify using the cut off property (2.18) to get

$$
\begin{aligned}
& 9\sum_{j=1}^{\infty}\int_0^T \left\|(\mathcal{S}_R-\mathcal{S}_{R'})\Phi e_j\right\|_{\mathrm{L}^2}^2 \mathrm{d}t + 9\int_0^T\int_{\mathrm{Z}}\left\|(\mathcal{S}_R-\mathcal{S}_{R'})\gamma(t,z)\right\|_{\mathrm{L}^2}^2\lambda(\mathrm{d}z)\,\mathrm{d}t \\
&\quad \leqslant \frac{9C}{R^{\varepsilon}}\sum_{j=1}^{\infty}\int_0^T \|\Phi e_j\|_{\mathrm{H}^{\varepsilon}}^2\mathrm{d}t + \frac{9C}{R^{\varepsilon}}\int_0^T\int_{\mathrm{Z}}\left\|\gamma(t,z)\right\|_{\mathrm{H}^{\varepsilon}}^2\lambda(\mathrm{d}z)\,\mathrm{d}t \\
&\quad \leqslant \frac{9C}{R^{\varepsilon}}\sum_{j=1}^{\infty}\int_0^T \|\Phi e_j\|_{\mathrm{H}^{s}}^2\mathrm{d}t + \frac{9C}{R^{\varepsilon}}\int_0^T\int_{\mathrm{Z}}\left\|\gamma(t,z)\right\|_{\mathrm{H}^{s}}^2\lambda(\mathrm{d}z)\,\mathrm{d}t = \frac{9C(K_2T+K_4)}{R^{\varepsilon}},
\end{aligned}
\tag{3.13}
$$

for $0<\varepsilon<s-2$. Hence from (3.11), we have

$$
\begin{aligned}
& \mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\left\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\right\|_{\mathrm{L}^2}^2\Big) \\
&\quad \leqslant \frac{2C(\mathbb{E}(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2)+N_1^2T+9(K_2T+K_4))}{R^{\varepsilon}} \\
&\qquad + \left(\frac{2CN_1^2}{R^{\varepsilon}}+4CN_1\right)\int_0^T\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t\wedge\tau_N}\left\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(r)\right\|_{\mathrm{L}^2}^2\Big)\,\mathrm{d}t,
\end{aligned}
\tag{3.14}
$$

where $\widetilde{\tau}_N := T\wedge\tau_N = T\wedge\tau_{N_1}\wedge\tau_{N_2}$. An application of the Gronwall's inequality in (3.14) yields

$$
\begin{aligned}
& \mathbb{E}\Big(\sup_{0\leqslant t\leqslant\widetilde{\tau}_N}\left\|\big(\mathbf{u}^R-\mathbf{u}^{R'}\big)(t)\right\|_{\mathrm{L}^2}^2\Big) \\
&\quad \leqslant \left(\frac{2C(\mathbb{E}(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2)+N_1^2T+9(K_2T+K_4))}{R^{\varepsilon}}\right)\exp\left\{\left(\frac{2CN_1^2}{R^{\varepsilon}}+4CN_1\right)T\right\}.
\end{aligned}
\tag{3.15}
$$

Note that the right hand side of the inequality (3.15) is independent of N_2 and on passing $N_2\to\infty$ yields $\widetilde{\tau}_N = T\wedge\tau_{N_1}\wedge\tau_{N_2}\to T\wedge\tau_{N_1}$. On passing $N_2, R, R'\to\infty$ and applying the dominated convergence theorem, one can easily see that the right hand side of the inequality (3.15) goes to zero and hence the sequence of solutions defined by $(\mathbf{u}^R, T\wedge\tau_N)$ (by redefining $N_1=N$) is Cauchy (as $R\to\infty$) in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{L}^2(\mathbb{R}^n)))$, where τ_N is the stopping time defined in (2.67). □

Proposition 3.2. *Let $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathrm{H}^s(\mathbb{R}^n))$ be $\mathscr{F}_0$-measurable for $s>n/2+1$. Then, the family of local strong solutions $(\mathbf{u}^R, T\wedge\tau_N)\to(\mathbf{u}, T\wedge\tau_N)$ strongly in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^{s'}(\mathbb{R}^n)))$ for any $s'<s$.*

Proof. It follows from (3.15) that $(\mathbf{u}^R, T\wedge\tau_N)\to(\mathbf{u}, T\wedge\tau_N)$ strongly in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N,\mathrm{L}^2(\mathbb{R}^n)))$. By using the Sobolev's interpolation inequality (Theorem 9.6, Remark 9.1, [27] with exponents $\frac{s}{s-s'}$ and

$\frac{s}{s'}$) and Hölder's inequality for $0 < s' < s$, we have

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|f\|_{\mathrm{H}^{s'}}^2\Big] \leqslant C_s\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|f\|_{\mathrm{L}^2}^{2(1-s'/s)}\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|f\|_{\mathrm{H}^s}^{2s'/s}\Big]$$
$$\leqslant C_s\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|f\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|f\|_{\mathrm{H}^s}^2\Big]\Big\}^{s'/s}. \tag{3.16}$$

Let us take $f = \mathbf{u}^R - \mathbf{u}$ in (3.16) to obtain

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^{s'}}^2\Big]$$
$$\leqslant C_s\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^s}^2\Big]\Big\}^{s'/s}$$
$$\leqslant C_s\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R\big\|_{\mathrm{H}^s}^2+\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}\|_{\mathrm{H}^s}^2\Big]\Big\}^{s'/s}$$
$$\leqslant C_s\big[2C\big(N,T,K_2,K_4,\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]\big)\big]^{s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\to 0, \tag{3.17}$$

as $R \to \infty$. Combining Proposition 2.11 and Proposition 3.1 and using the Sobolev's interpolation yields $(\mathbf{u}^R, T\wedge\tau_N) \to (\mathbf{u}, T\wedge\tau_N)$ strongly in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'}(\mathbb{R}^n)))$ for any $s' < s$. $\square$

Remark 3.3. Since $\mathbf{u}^R \to \mathbf{u}$ strongly in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'}(\mathbb{R}^n)))$, we get $\nabla\cdot\mathbf{u}^R \to \nabla\cdot\mathbf{u}$ in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$, for any $s' > n/2+1$ and hence $\nabla\cdot\mathbf{u}^R = 0$ implies $\nabla\cdot\mathbf{u} = 0$.

Next let us prove a simple estimate to deal with the nonlinear terms.

Lemma 3.4. *Fix $s > n/2$ and let $\mathbf{u}, \mathbf{w} \in \mathrm{H}^s$ with $\nabla\cdot\mathbf{v} = 0$. Then we have*

$$\big\|(\mathbf{v}\cdot\nabla)\mathbf{w}\big\|_{\mathrm{H}^{s-1}} \leqslant C\|\mathbf{v}\|_{\mathrm{H}^s}\|\mathbf{w}\|_{\mathrm{H}^s}.$$

Proof. Since, $\nabla\cdot\mathbf{v} = 0$, we have $(\mathbf{v}\cdot\nabla)\mathbf{w} = \nabla\cdot(\mathbf{v}\otimes\mathbf{w})$. For $s > n/2$, H^{s-1} is an algebra and hence

$$\big\|(\mathbf{v}\cdot\nabla)\mathbf{w}\big\|_{\mathrm{H}^{s-1}} = \big\|\nabla\cdot(\mathbf{v}\otimes\mathbf{w})\big\|_{\mathrm{H}^{s-1}} \leqslant C\|\mathbf{v}\otimes\mathbf{w}\|_{\mathrm{H}^s} \leqslant C\|\mathbf{v}\|_{\mathrm{H}^s}\|\mathbf{w}\|_{\mathrm{H}^s}. \tag{3.18}$$

$\square$

Proposition 3.5. *Let τ_N be the stopping time defined in (2.67), then for $s' > n/2+1$, the nonlinear term $\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]$ converges to $(\mathbf{u}\cdot\nabla)\mathbf{u}$ strongly in $\mathrm{L}^1(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$ as $R \to \infty$.*

Proof. For $s' > n/2+1$, by using (2.16), (2.17), Lemma 3.4 and Hölder's inequality, for $0 < \varepsilon < 1$, we obtain

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathcal{S}_R\big[\big(\mathbf{u}^R\cdot\nabla\big)\mathbf{u}^R\big]-(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|_{\mathrm{H}^{s'-1}}\Big]$$
$$\leqslant \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathcal{S}_R\big[\big(\big(\mathbf{u}^R-\mathbf{u}\big)\cdot\nabla\big)\mathbf{u}^R\big]\big\|_{\mathrm{H}^{s'-1}}\Big]+\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathcal{S}_R\big[(\mathbf{u}\cdot\nabla)\big(\mathbf{u}^R-\mathbf{u}\big)\big]\big\|_{\mathrm{H}^{s'-1}}\Big]$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathcal{S}_R\big[(\mathbf{u}\cdot\nabla)\mathbf{u}\big]-(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|_{\mathrm{H}^{s'-1}}\Big]\\
&\leqslant C\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\big((\mathbf{u}^R-\mathbf{u})\cdot\nabla\big)\mathbf{u}^R\big\|_{\mathrm{H}^{s'-1}}\Big]+C\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|(\mathbf{u}\cdot\nabla)\big(\mathbf{u}^R-\mathbf{u}\big)\big\|_{\mathrm{H}^{s'-1}}\Big]\\
&\quad+\frac{C}{R^\varepsilon}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|_{\mathrm{H}^{s'-1+\varepsilon}}\Big]\\
&\leqslant C\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big(\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^{s'}}\big\|\mathbf{u}^R\big\|_{\mathrm{H}^{s'}}\big)\Big]+C\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big(\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^{s'}}\|\mathbf{u}\|_{\mathrm{H}^{s'}}\big)\Big]\\
&\quad+\frac{C}{R^\varepsilon}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}\|_{\mathrm{H}^{s'+\varepsilon}}\|\mathbf{u}\|_{\mathrm{H}^{s'+\varepsilon}}\Big]\\
&\leqslant C\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^{s'}}^2\Big)^{1/2}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R\big\|_{\mathrm{H}^{s'}}^2\Big)^{1/2}\\
&\quad+C\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R-\mathbf{u}\big\|_{\mathrm{H}^{s'}}^2\Big)^{1/2}\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}\|_{\mathrm{H}^{s'}}^2\Big)^{1/2}\\
&\quad+\frac{C}{2R^\varepsilon}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}\|_{\mathrm{H}^{s'+\varepsilon}}^2\Big]+\frac{C}{2R^\varepsilon}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}\|_{\mathrm{H}^{s'+\varepsilon}}^2\Big]\\
&\to 0\quad\text{as } R\to\infty,
\end{aligned}\tag{3.19}
$$

since $(\mathbf{u}^R, T\wedge\tau_N)\to(\mathbf{u}, T\wedge\tau_N)$ strongly in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^{s'}(\mathbb{R}^n)))$ and for $s>s'>n/2+1$, $\mathbf{u}^R,\mathbf{u}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^s(\mathbb{R}^n)))$. Hence, for the stopping time τ_N defined in (2.67), we have $\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]\to(\mathbf{u}\cdot\nabla)\mathbf{u}$ strongly in $\mathrm{L}^1(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^{s'-1}(\mathbb{R}^n)))$. $\square$

Remark 3.6. From Proposition 3.5, it can be easily seen that $\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]\to(\mathbf{u}\cdot\nabla)\mathbf{u}$ strongly in $\mathrm{L}^1(\Omega;\mathrm{L}^2(0,T\wedge\tau_N;\mathrm{H}^{s'-1}(\mathbb{R}^n)))$, since

$$
\mathrm{L}^1\big(\Omega;\mathrm{L}^\infty\big(0,T\wedge\tau_N;\mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big)\subset\subset\mathrm{L}^1\big(\Omega;\mathrm{L}^2\big(0,T\wedge\tau_N;\mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big),
$$

for the stopping time τ_N defined in (2.67).

Proposition 3.7. *Let $\mathbf{u}_0$ be $\mathscr{F}_0$-measurable and $\mathbb{E}[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2]<\infty$ for $s>n/2+1$. Then, for any $s'>n/2+1$ and $s'<s$, $\mathcal{S}_R\mathbf{u}_0\to\mathbf{u}_0$ as $R\to\infty$ in $\mathrm{L}^2(\Omega;\mathrm{H}^{s'-1}(\mathbb{R}^n))$.*

Proof. By using (2.17), for $0<\varepsilon<1$, we have

$$
\mathbb{E}\big[\|\mathcal{S}_R\mathbf{u}_0-\mathbf{u}_0\|_{\mathrm{H}^{s'-1}}^2\big]\leqslant\frac{C}{R^\varepsilon}\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^{s'-1+\varepsilon}}^2\big]\leqslant\frac{C}{R^\varepsilon}\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]\to 0,
$$

as $R\to\infty$. $\square$

Proposition 3.8. *Let τ_N be the stopping time defined in (2.67), then for any $s' > n/2+1$ and $t \in [0, T\wedge\tau_N)$,*

$$\mathcal{S}_R\bigg(\int_0^t \Phi\, \mathrm{dW}(s)\bigg) \to \int_0^t \Phi\, \mathrm{dW}(s), \quad \textit{as } R\to\infty,$$

in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$.

Proof. By using (2.17) and the Burkholder–Davis–Gundy inequality, for $0<\varepsilon<1$ and $T>0$, we get

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t \Phi\, \mathrm{dW}(s)\bigg) - \int_0^t \Phi\, \mathrm{dW}(s)\bigg\|^2_{\mathrm{H}^{s'-1}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \Phi\, \mathrm{dW}(s)\bigg\|^2_{\mathrm{H}^{s'-1+\varepsilon}}\bigg] \leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \Phi\, \mathrm{dW}(s)\bigg\|^2_{\mathrm{H}^{s}}\bigg] \\
&\quad= \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \mathrm{J}^s\Phi\, \mathrm{dW}(s)\bigg\|^2_{\mathrm{L}^2}\bigg] \leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\mathrm{Tr}\big(\big(\mathrm{J}^s\Phi\big)^*\big(\mathrm{J}^s\Phi\big)\big)\,\mathrm{d}t\bigg] \\
&\quad\leqslant \frac{CT\,\mathrm{Tr}((\mathrm{J}^s\Phi)^*(\mathrm{J}^s\Phi))}{R^\varepsilon} \to 0,
\end{aligned}$$

as $R\to\infty$. $\square$

Proposition 3.9. *Let τ_N be the stopping time defined in (2.67), then for any $s' > n/2+1$ and $t \in [0, T\wedge\tau_N)$,*

$$\mathcal{S}_R\bigg(\int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\bigg) \to \int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z), \quad \textit{as } R\to\infty,$$

in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$.

Proof. By using (2.17) and the Burkholder–Davis–Gundy inequality, for $0<\varepsilon<1$ and $T>0$, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\bigg) - \int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1+\varepsilon}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t\int_{\mathrm{Z}} \gamma(s-,z)\widetilde{\mathscr{N}}(\mathrm{d}s,\mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\int_{\mathrm{Z}}\|\gamma(t,z)\|^2_{\mathrm{H}^s}\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg] \leqslant \frac{C}{R^\varepsilon}\int_0^T\int_{\mathrm{Z}}\|\gamma(t,z)\|^2_{\mathrm{H}^s}\lambda(\mathrm{d}z)\,\mathrm{d}t = \frac{CK_4}{R^\varepsilon} \to 0,
\end{aligned}$$

as $R\to\infty$. $\square$

Remark 3.10. Since

$$\mathrm{L}^2\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big) \subset\subset \mathrm{L}^1\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big)$$

and

$$\mathrm{L}^1\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big) \subset\subset \mathrm{L}^1\big(\Omega; \mathrm{L}^2\big(0, T \wedge \tau_N; \mathrm{H}^{s'-1}\big(\mathbb{R}^n\big)\big)\big),$$

for the stopping time τ_N defined in (2.67), we have

$$\mathcal{S}_R\bigg(\int_0^t \Phi\,\mathrm{dW}(s)\bigg) \to \int_0^t \Phi\,\mathrm{dW}(s), \quad \text{as } R \to \infty \quad \text{and}$$
$$\mathcal{S}_R\bigg(\int_0^t \int_{\mathrm{Z}} \gamma(s-, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg) \to \int_0^t \int_{\mathrm{Z}} \gamma(s-, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \quad \text{as } R \to \infty,$$

for $t \in [0, T \wedge \tau_N)$, in $\mathrm{L}^1(\Omega; \mathrm{L}^2(0, T \wedge \tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$.

Theorem 3.11. *Let τ_N be the stopping time defined in (2.67) and let $\mathbf{u}_0 \in \mathrm{L}^2(\Omega; \mathrm{H}^s(\mathbb{R}^n))$ be $\mathscr{F}_0$-measurable for $s > n/2+1$. Then there exists a local in time strong solution of problem (2.7)–(2.8) such that*

(i) $\mathbf{u} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$,
(ii) *the $\mathscr{F}_t$-adapted paths of* $(\mathbf{u}, T \wedge \tau_N)$ *are càdlàg.*

Proof. From the above construction, we can easily pass the limit as $R \to \infty$ in the equation

$$\mathbf{u}^R(t) = \mathcal{S}_R\mathbf{u}_0 - \int_0^t \mathcal{S}_R\mathrm{P}_{\mathcal{H}}\big[\big(\mathbf{u}^R \cdot \nabla\big)\mathbf{u}^R\big]\,\mathrm{d}s + \mathcal{S}_R\int_0^t \Phi\,\mathrm{dW}(s) + \mathcal{S}_R\int_0^t \int_{\mathrm{Z}} \gamma(s-, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z),$$

for any $t \in [0, T \wedge \tau_N)$, so that $(\mathbf{u}, T \wedge \tau_N)$ solves the stochastic Euler equations (2.7)–(2.8) in $\mathrm{L}^1(\Omega; \mathrm{L}^2(0, T \wedge \tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$, for $s' > n/2 + 1$. From Proposition 3.1, Remark 3.6, Proposition 3.7 and Remark 3.10, we conclude that $(\mathbf{u}, T \wedge \tau_N)$ solves the system (2.7)–(2.8) as an equality in $\mathrm{L}^1(\Omega; \mathrm{L}^2(0, T \wedge \tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$, for $s' > n/2+1$.

Let us now apply Banach–Alaoglu theorem (Theorem 4.18, [40]) to the Fourier truncated sequence $(\mathbf{u}^R, T \wedge \tau_N)$, the solution of the truncated stochastic Euler equations (2.25)–(2.26). From Proposition 2.11, the sequence $\mathbf{u}^R$ is uniformly bounded in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$ and the bound is independent of R for the stopping time τ_N defined in (2.67). Since $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$ is the dual of $\mathrm{L}^2(\Omega; \mathrm{L}^1(0, T \wedge \tau_N; \mathrm{H}^{-s}(\mathbb{R}^n)))$, the separability of $\mathrm{L}^2(\Omega; \mathrm{L}^1(0, T \wedge \tau_N; \mathrm{H}^{-s}(\mathbb{R}^n)))$ (Remark 10.1.10 and Theorem 10.1.13, [38]) and the uniform bounds for $\mathbf{u}^R$ in $[0, T \wedge \tau_N)$ guarantee the existence of subsequence $\mathbf{u}^{R^m}$ such that

$$\mathbf{u}^{R^m} \xrightarrow{w^*} \mathbf{u} \quad \text{in } \mathrm{L}^2\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^s\big(\mathbb{R}^n\big)\big)\big).$$

From the above weak star convergence, the limit $\mathbf{u}$ satisfies

$$\mathbf{u} \in \mathrm{L}^2\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^s\big(\mathbb{R}^n\big)\big)\big)$$

and $(\mathbf{u}, T \wedge \tau_N)$ solves (2.7)–(2.8) for $s > n/2 + 1$. From Proposition 3.1, $(\mathbf{u}^R, T \wedge \tau_N)$ is almost surely uniformly convergent on finite intervals $[0, T \wedge \tau_N)$ to $(\mathbf{u}, T \wedge \tau_N)$, from which it follows that $(\mathbf{u}, T \wedge \tau_N)$ is adapted and càdlàg (Theorem 6.2.3, [1]). $\square$

Remark 3.12. From the existence of local strong solution $(\mathbf{u}, T \wedge \tau_N)$ to the stochastic Euler equations (2.7)–(2.8), one can easily see that

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N} \|\mathbf{u}(t)\|_{\mathrm{H}^s}^2\Big] \leqslant \big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big] + 18(K_2 T + K_4)\big)e^{4NT} < \infty, \tag{3.20}$$

for the stopping time τ_N defined in (2.15) and for any $T > 0$.

Theorem 3.13. *Let $0 < \delta < 1$ be given. Then, for the stopping time τ_N defined in (2.15), we have*

$$\mathbb{P}\{\tau_N > \delta\} \geqslant 1 - C\delta^2\big\{\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big] + 18\big(K_2\delta + K_4(\delta)\big)\big)\big\}, \tag{3.21}$$

where C is a positive constant independent of δ.

Theorem 3.13 can be proved in the same way as that of Theorem 2.13.

Theorem 3.14. *Let τ_N be the stopping time defined in (2.15) and let $\mathbf{u}_0 \in \mathrm{L}^2(\Omega; \mathrm{H}^s(\mathbb{R}^n))$ be $\mathscr{F}_0$-measurable for $s > n/2 + 1$. Let $(\mathbf{u}_j, T \wedge \tau_N)$, $j = 1, 2$ be two $\mathscr{F}_t$-adapted processes with càdlàg paths that are local strong solutions of (2.7)–(2.8) having same initial value $\mathbf{u}_j(0) = \mathbf{u}_0$, such that*

$$\mathbf{u}_j \in \mathrm{L}^2\big(\Omega; \mathrm{L}^\infty\big(0, T \wedge \tau_N; \mathrm{H}^s\big(\mathbb{R}^n\big)\big)\big),$$

for $s > n/2 + 1$. Then $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for any $t > 0$, as functions in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{L}^2(\mathbb{R}^n)))$.

Proof. Let τ_N be the stopping time defined in (2.15) and let $(\mathbf{u}_1, T \wedge \tau_N)$ and $(\mathbf{u}_2, T \wedge \tau_N)$ be two local strong solutions of the system of Eqs (2.7)–(2.8) having common initial data $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$ such that $\mathbb{E}(\|\mathbf{u}_0\|_{\mathrm{H}^s}^2) < \infty$. Let us take the difference between the two equations satisfied by $(\mathbf{u}_1, T \wedge \tau_N)$ and $(\mathbf{u}_2, T \wedge \tau_N)$ to obtain

$$\mathrm{d}(\mathbf{u}_1 - \mathbf{u}_2) = -\mathrm{P}_{\mathcal{H}}\big[(\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2\big]\,\mathrm{d}t. \tag{3.22}$$

Let us apply Itô's formula to $\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2}^2$ to find

$$\big\|(\mathbf{u}_1 - \mathbf{u}_2)(t \wedge \tau_N)\big\|_{\mathrm{L}^2}^2 = -2\int_0^{t\wedge\tau_N} \big((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\big)_{\mathrm{L}^2}\,\mathrm{d}s. \tag{3.23}$$

Now we take the nonlinear term, and use the divergence free condition and Hölder's inequality to get

$$\begin{aligned}
&\big|-\big((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\big)_{\mathrm{L}^2}\big| \\
&\quad = \big|-\big[\big(\big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla\big)\mathbf{u}_1, (\mathbf{u}_1 - \mathbf{u}_2)\big)_{\mathrm{L}^2} + \big(\big((\mathbf{u}_2 \cdot \nabla)(\mathbf{u}_1 - \mathbf{u}_2)\big), (\mathbf{u}_1 - \mathbf{u}_2)\big)_{\mathrm{L}^2}\big]\big| \\
&\quad \leqslant \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2}\|\nabla\mathbf{u}_1\|_{\mathrm{L}^\infty}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2} \leqslant CN\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2}^2.
\end{aligned} \tag{3.24}$$

An application of (3.24) in (3.23) yields

$$\big\|(\mathbf{u}_1-\mathbf{u}_2)(t\wedge\tau_N)\big\|_{\mathrm{L}^2}^2 \leqslant CN\int_0^{t\wedge\tau_N}\big\|(\mathbf{u}_1-\mathbf{u}_2)(s)\big\|_{\mathrm{L}^2}^2\,\mathrm{d}s, \tag{3.25}$$

Let us take expectation in (3.25) to get

$$\mathbb{E}\Big[\big\|(\mathbf{u}_1-\mathbf{u}_2)(t\wedge\tau_N)\big\|_{\mathrm{L}^2}^2\Big] \leqslant CN\int_0^{t}\mathbb{E}\Big[\big\|(\mathbf{u}_1-\mathbf{u}_2)(s\wedge\tau_N)\big\|_{\mathrm{L}^2}^2\Big]\,\mathrm{d}s. \tag{3.26}$$

An application of the Gronwall's inequality in (3.26) yields

$$\mathbb{E}\Big[\big\|(\mathbf{u}_1-\mathbf{u}_2)(t\wedge\tau_N)\big\|_{\mathrm{L}^2}^2\Big] \leqslant 0, \tag{3.27}$$

for all $0\leqslant t\leqslant T$. Hence, we get $\mathbf{u}_1(t\wedge\tau_N)=\mathbf{u}_2(t\wedge\tau_N)$, a.s., for all $0\leqslant t\leqslant T$. In a similar way, one can prove that

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|(\mathbf{u}_1-\mathbf{u}_2)(t)\big\|_{\mathrm{L}^2}^2\Big] \leqslant CN\int_0^{T}\mathbb{E}\Big[\sup_{0\leqslant s\leqslant t\wedge\tau_N}\big\|(\mathbf{u}_1-\mathbf{u}_2)(s)\big\|_{\mathrm{L}^2}^2\Big]\,\mathrm{d}t, \tag{3.28}$$

for any $T>0$ and then we obtain

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|(\mathbf{u}_1-\mathbf{u}_2)(t)\big\|_{\mathrm{L}^2}^2\Big] \leqslant 0. \tag{3.29}$$

Hence, we get $\mathbf{u}_1(t)=\mathbf{u}_2(t)$, a.s., for any $t>0$ as functions in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{L}^2(\mathbb{R}^n)))$. □

Theorem 3.15. *Let τ_N be the stopping time defined in (2.15) and let $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathrm{H}^s(\mathbb{R}^n))$ be $\mathscr{F}_0$-measurable for $s>n/2+1$. Let $(\mathbf{u}_j, T\wedge\tau_N)$, $j=1,2$ be two $\mathscr{F}_t$-adapted processes with càdlàg paths that are local strong solutions of (2.7)–(2.8) having same initial data $\mathbf{u}_j(0)=\mathbf{u}_0$, such that*

$$\mathbf{u}_j\in\mathrm{L}^2\big(\Omega;\mathrm{L}^\infty\big(0,T\wedge\tau_N;\mathrm{H}^s\big(\mathbb{R}^n\big)\big)\big),$$

for $s>n/2+1$. Then $\mathbf{u}_1(t)=\mathbf{u}_2(t)$, a.s., for any $t>0$ as functions in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T\wedge\tau_N;\mathrm{H}^{s'}(\mathbb{R}^n)))$, for any $0<s'<s$.

Proof. Using the Sobolev's interpolation theorem and Hölder's inequality for $0<s'<s$, we find

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathrm{H}^{s'}}^2\Big]\\
&\quad\leqslant C_s\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathrm{H}^s}^2\Big]\Big\}^{s'/s}\\
&\quad\leqslant C_s\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1\|_{\mathrm{H}^s}^2+\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_2\|_{\mathrm{H}^s}^2\Big]\Big\}^{s'/s}\\
&\quad\leqslant C_s\big[2C\big(N,K,T,\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big]\big)\big]^{s'/s}\Big\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathrm{L}^2}^2\Big]\Big\}^{1-s'/s}=0,
\end{aligned}\tag{3.30}$$

for any $T > 0$, since $\mathbf{u}_1(\cdot) = \mathbf{u}_2(\cdot)$, a.s., in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{L}^2(\mathbb{R}^n)))$. Hence we get $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for any $t > 0$ in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^{s'}(\mathbb{R}^n)))$ for any $0 < s' < s$. □

Theorem 3.14 and Theorem 3.15 prove the uniqueness of strong solution $(\mathbf{u}, T \wedge \tau_N)$ of the stochastic Euler equations perturbed by Lévy noise. Now if we take $(\mathbf{u}_1, T \wedge \tau_{N_1})$ and $(\mathbf{u}_2, T \wedge \tau_{N_2})$ are two local strong solutions of (2.7)–(2.8), then from (3.24), we get

$$\left|-\big((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\big)_{\mathrm{L}^2}\right| \leqslant C N_1 \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2}^2. \tag{3.31}$$

Thus by Theorem 3.14, we obtain $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for all $t \in [0, T \wedge \tau_{N_1})$. A similar calculation to (3.24) also shows that

$$\left|-\big((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\big)_{\mathrm{L}^2}\right| \leqslant C N_2 \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{L}^2}^2, \tag{3.32}$$

and hence we find $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for all $t \in [0, T \wedge \tau_{N_2})$. Combining the two cases, once can easily see that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for all $t \in [0, T \wedge \tau_{N_1} \wedge \tau_{N_2})$ and hence $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, a.s., for all $t \in [0, \tau_{N_1} \wedge \tau_{N_2})$, since $T > 0$ is arbitrary. If both are maximal local strong solutions, then it is immediate that $\tau_{N_1} = \tau_{N_2}$, a.s. Since, if $(\mathbf{u}_1, \tau_{N_1})$ is a maximal local strong solution, then $\sup_{0 \leqslant t \leqslant \tau_{N_1}} \|\mathbf{u}_1(t)\|_{\mathrm{H}^s} = \infty$, and this must imply that $\tau_{N_2} > \tau_{N_1}$. Otherwise, $\sup_{0 \leqslant t \leqslant \tau_{N_2}} \|\mathbf{u}_1(t)\|_{\mathrm{H}^s} = \infty$, by using the equality of $\mathbf{u}_i(\cdot)$, for $i = 1, 2$ and the maximality of $\mathbf{u}_2(t)$ in $[0, \tau_{N_1} \wedge \tau_{N_2}) = [0, \tau_{N_2})$ and we arrive at a contradiction. A similar argument on $(\mathbf{u}_2, \tau_{N_2})$ imply $\tau_{N_2} < \tau_{N_1}$ and hence $\tau_{N_2} = \tau_{N_1}$.

Now, let $(\mathbf{u}_N, T \wedge \tau_N)$ be the unique local strong solution to the system (2.7)–(2.8) corresponding to the stopping time (2.15). Then, for $N_1 < N_2$, by using Theorem 3.14 and Theorem 3.15, we have

$$\mathbf{u}_{N_1}(t) = \mathbf{u}_{N_2}(t), \quad \text{a.s., for all } t \in [0, \tau_{N_1} \wedge \tau_{N_2}],$$

since $T > 0$ is arbitrary. Thus from the definition of stopping time (2.15), we have $\tau_{N_1} \leqslant \tau_{N_2}$, a.s. We can now define $\tau(\omega) := \lim_{N\to\infty} \tau_N(\omega)$, a.s., and

$$\mathbf{u}(t) := \begin{cases} \lim_{N\to\infty} \mathbf{u}_N(t), & \text{for } 0 \leqslant t < \tau, \\ 0, & \text{for } t \geqslant \tau, \end{cases}$$

a.s. Hence, $(\mathbf{u}, \tau)$ is a local strong solution (in fact maximal) to the problem (2.7)–(2.8) and for a given $0 < \delta < 1$, we have

$$\mathbb{P}\{\tau > \delta\} \geqslant 1 - C\delta^2\big\{\big(2\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\big] + 18\big(K_2\delta + K_4(\delta)\big)\big)\big\},$$

since $\{\tau_N > \delta\} \subset \{\tau > \delta\}$ (see Theorem 3.13 and Theorem 2.13). A characterization of the maximal solution for stochastic Euler system (2.7)–(2.8) is given in the next theorem (Theorem 3.16).

Theorem 3.16. *Assume that Theorem 3.11 and Theorem 3.14 hold. Then there exists a unique pair $(\mathbf{u}, \tau_\infty)$, which is a maximal strong solution of (2.7)–(2.8) such that*

$$\sup_{0 \leqslant s \leqslant \tau_\infty} \big\|\mathbf{u}(s)\big\|_{\mathrm{H}^s} = \infty, \tag{3.33}$$

on the set $\{\omega \in \Omega : \tau_\infty(\omega) < \infty\}$.

Proof. Let us denote by $\mathcal{L}$, the set of all stopping times such that $\tau \in \mathcal{L}$ if and only if there exists a process $\mathbf{u}(\cdot)$ such that $(\mathbf{u}, \tau)$ is a local strong solution of the stochastic Euler equations (2.7)–(2.8). It can be easily seen that

$$\tau_1, \tau_2 \in \mathcal{L} \Rightarrow \tau_1 \vee \tau_2, \tau_1 \wedge \tau_2 \in \mathcal{L}. \tag{3.34}$$

For each $k \in \mathbb{N}$, let us take $\tau_k \in \mathcal{L}$ such that $(\mathbf{u}_k, \tau_k)$ be the unique local strong solution of (2.7)–(2.8). Then for each τ_k, the process $\mathbf{u}_k(\cdot)$ having càdlàg paths such that $(\mathbf{u}_k, \tau_k)$ is a local strong solution of (2.7)–(2.8) with

$$\tau_k = \inf_{t\geqslant 0}\left\{t : \left\|\nabla \mathbf{u}_k(t)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}_k(t)\right\|_{\mathrm{H}^{s-1}} \geqslant k\right\} \wedge T, \quad k \in \mathbb{N}, \tag{3.35}$$

for some $T > 0$. Now for $n > k$, let us define a sequence of stopping times $\tau_{k,n}$ such that

$$\tau_{k,n} = \inf_{t\geqslant 0}\left\{t : \left\|\nabla \mathbf{u}_n(t)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}_n(t)\right\|_{\mathrm{H}^{s-1}} \geqslant k\right\} \wedge T, \quad k, n \in \mathbb{N}. \tag{3.36}$$

Since

$$\tau_n = \inf_{t\geqslant 0}\left\{t : \left\|\nabla \mathbf{u}_n(t)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}_n(t)\right\|_{\mathrm{H}^{s-1}} \geqslant n\right\} \wedge T, \quad n \in \mathbb{N}, \tag{3.37}$$

it is clear from the definition of τ_n that $\tau_{k,n} \leqslant \tau_n$, a.s., for $n > k$. Thus $(\mathbf{u}_n, \tau_{k,n})$ is a local strong solution of (2.7)–(2.8). But $(\mathbf{u}_k, \tau_k)$ is also a local strong solution of (2.7)–(2.8). Hence from the uniqueness theorem (Theorem 3.14, Theorem 3.15), we get $\mathbf{u}_k(t) = \mathbf{u}_n(t)$, a.s., for all $t \in [0, \tau_k \wedge \tau_{k,n})$. This proves that $\mathbf{u}_k(t) = \mathbf{u}_n(t)$, a.s., for all $t \in [0, \tau_k)$ and hence $\tau_k < \tau_n$, a.s., for all $k < n$. Thus $\{\tau_k : k \in \mathbb{N}\}$ is an increasing sequence in $\mathcal{L}$ and hence it has a limit in $\mathcal{L}$ (Proposition 3.9, [6]). Let us denote the limit by $\tau_\infty := \lim_{k\to\infty}\tau_k$. By letting $k \to \infty$, let $\{\mathbf{u}(t), 0 \leqslant t < \tau_\infty\}$ be the stochastic process defined by

$$\mathbf{u}(t) = \mathbf{u}_k(t), \quad t \in [\tau_{k-1}, \tau_k), k \geqslant 1, \tag{3.38}$$

where $\tau_0 = 0$. By making use of uniqueness results, we have $\mathbf{u}(t \wedge \tau_k) = \mathbf{u}_k(t \wedge \tau_k)$ for any $t > 0$. As $k \to \infty$, we are thus justified to define a process $(\mathbf{u}, \tau_\infty)$ such that $(\mathbf{u}, \tau_\infty)$ is a local strong solution of (2.7)–(2.8) on the set $\{\omega : \tau_\infty(\omega) < T\}$ and hence we have

$$\begin{aligned}
\lim_{t\uparrow\tau_\infty}\left[\|\mathbf{u}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^s)}\right] &= \lim_{t\uparrow\tau_\infty}\left[\sup_{0\leqslant s\leqslant t}\left\|\mathbf{u}(s)\right\|_{\mathrm{H}^s}\right] \geqslant \lim_{k\uparrow\infty}\left[\sup_{0\leqslant s\leqslant\tau_k}\left\|\mathbf{u}(s)\right\|_{\mathrm{H}^s}\right] = \lim_{k\uparrow\infty}\left[\sup_{0\leqslant s\leqslant\tau_k}\left\|\mathbf{u}_k(s)\right\|_{\mathrm{H}^s}\right] \\
&\geqslant \mathcal{K}\lim_{k\uparrow\infty}\left[\sup_{0\leqslant s\leqslant\tau_k}\left(\left\|\nabla\mathbf{u}_k(s)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}_k(s)\right\|_{\mathrm{H}^{s-1}}\right)\right] = \infty,
\end{aligned} \tag{3.39}$$

where $\mathcal{K}$ is defined in (2.71).

Let us now prove that the maximal solution obtained above is unique. Let us assume that the pair $(\mathbf{u}_1, \sigma_\infty)$ be another maximal solution. Here $\{\sigma_k, k \geqslant 0\}$ is an increasing sequence of stopping times converging to σ_∞ and is defined by

$$\sigma_k = \inf_{t\geqslant 0}\left\{t : \left\|\nabla \mathbf{u}_1(t)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}_1(t)\right\|_{\mathrm{H}^{s-1}} \geqslant k\right\} \wedge T, \quad k \in \mathbb{N}. \tag{3.40}$$

By a similar argument above and by the uniqueness theorem (Theorem 3.14 and Theorem 3.15) one can prove that $\mathbf{u}(t)=\mathbf{u}_1(t)$, a.s., for all $t\in[0,\tau_k\wedge\sigma_k]$, for $k\geqslant 0$. Let us take $k\uparrow\infty$ so that we get

$$\mathbf{u}(t)=\mathbf{u}_1(t), \qquad \text{a.s., for all } t\in[0,\tau_\infty\wedge\sigma_\infty]. \tag{3.41}$$

From (3.41), we can easily conclude that $\tau_\infty=\sigma_\infty$, a.s. If $\tau_\infty\neq\sigma_\infty$, then either $\tau_\infty>\sigma_\infty$ or $\tau_\infty<\sigma_\infty$. In the first case, we have

$$\begin{aligned}
\lim_{t\uparrow\tau_\infty}\big[\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^s)}\big] &= \lim_{t\uparrow\tau_\infty}\Big[\sup_{0\leqslant s\leqslant t}\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}(s)\big\|_{\mathrm{H}^s}\Big]\\
&\geqslant \mathcal{K}\lim_{t\uparrow\tau_\infty}\Big[\sup_{0\leqslant s\leqslant t}\big(\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\nabla\mathbf{u}(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \mathcal{K}\lim_{k\uparrow\infty}\Big[\sup_{0\leqslant s\leqslant \tau_k}\big(\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\nabla\mathbf{u}(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \mathcal{K}\lim_{k\uparrow\infty}\Big[\sup_{0\leqslant s\leqslant \sigma_k}\big(\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\nabla\mathbf{u}_1(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}_1(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \infty.
\end{aligned} \tag{3.42}$$

For $\tau_\infty<\sigma_\infty$, we have

$$\begin{aligned}
\lim_{t\uparrow\tau_\infty}\big[\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}_1\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^s)}\big] &= \lim_{t\uparrow\tau_\infty}\Big[\sup_{0\leqslant s\leqslant t}\big\|\mathbf{1}_{\{\sigma_\infty<\tau_\infty\}}\mathbf{u}_1(s)\big\|_{\mathrm{H}^s}\Big]\\
&\geqslant \mathcal{K}\lim_{t\uparrow\sigma_\infty}\Big[\sup_{0\leqslant s\leqslant t}\big(\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\nabla\mathbf{u}_1(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\mathbf{u}_1(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \mathcal{K}\lim_{k\uparrow\infty}\Big[\sup_{0\leqslant s\leqslant \sigma_k}\big(\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\nabla\mathbf{u}_1(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\mathbf{u}_1(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \mathcal{K}\lim_{k\uparrow\infty}\Big[\sup_{0\leqslant s\leqslant \tau_k}\big(\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\nabla\mathbf{u}(s)\big\|_{\mathrm{L}^\infty}+\big\|\mathbf{1}_{\{\sigma_\infty>\tau_\infty\}}\mathbf{u}(s)\big\|_{\mathrm{H}^{s-1}}\big)\Big]\\
&= \infty.
\end{aligned} \tag{3.43}$$

The first identity (3.42) contradicts the fact that $\mathbf{u}(\cdot)$ does not explode before the time τ_∞ and the second identity contradicts the fact that $\mathbf{u}_1(\cdot)$ does not explode before the time σ_∞. Hence, we must have $\tau_\infty=\sigma_\infty$, a.s., and this proves the uniqueness of the maximal local strong solution $(\mathbf{u},\tau_\infty)$ of the stochastic Euler equations (2.7)–(2.8). $\square$

Similar ideas of proving maximal local solutions for modeling the flow of liquid crystals can be found in Proposition 3.11, [6] and viscous hydrodynamic systems with multiplicative type of jump noise in Theorem 3.5, [3].

4. Stochastic Euler equations with multiplicative Lévy noise

The stochastic Euler equations perturbed by multiplicative Lévy noise in $(0,T)$ (after taking the Helmholtz–Hodge orthogonal projection $\mathrm{P}_{\mathcal{H}}$) can be written in the Itô stochastic differential form as

$$\mathrm{d}\mathbf{u}(t) = -\mathrm{P}_{\mathcal{H}}\big(\mathbf{u}(t)\cdot\nabla\big)\mathbf{u}(t)\,\mathrm{d}t + \Phi\big(\mathbf{u}(t)\big)\,\mathrm{dW}(t) + \int_{\mathrm{Z}} \gamma\big(\mathbf{u}(t-), z\big)\widetilde{\mathscr{N}}(\mathrm{d}t,\mathrm{d}z), \tag{4.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{4.2}$$

where $\mathbf{u}_0 \in \mathrm{L}^2(\Omega; \mathrm{H}^s(\mathbb{R}^n))$ for $s > n/2 + 1$ with $\nabla\cdot\mathbf{u}_0 = 0$ and $(x,t,\omega) \in \mathbb{R}^n \times [0,T] \times \Omega$, $n = 2, 3$. We need some additional assumptions on the noise coefficient to prove the existence and uniqueness of local strong solutions to the system (4.1)–(4.2). Let $\mathscr{L}_2(\mathcal{H}, \mathrm{H}^s)$ be the space of all Hilbert–Schmidt operators from $\mathcal{H}$ to H^s ([16]). For an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in $\mathcal{H}$, we know that

$$\begin{aligned}\mathrm{Tr}\big(\big(\Phi(\mathbf{u})\big)^*\Phi(\mathbf{u})\big) &= \sum_{j=1}^{\infty}\big(\big(\Phi(\mathbf{u})\big)^*\Phi(\mathbf{u})e_j, e_j\big)_{\mathrm{L}^2} = \sum_{j=1}^{\infty}\big(\Phi(\mathbf{u})e_j, \Phi(\mathbf{u})e_j\big)_{\mathrm{L}^2} \\ &= \sum_{j=1}^{\infty}\big\|\Phi(\mathbf{u})e_j\big\|_{\mathrm{L}^2}^2 = \big\|\Phi(\mathbf{u})\big\|_{\mathscr{L}_2(\mathcal{H},\mathcal{H})}^2.\end{aligned} \tag{4.3}$$

Let us assume that the noise coefficient $\Phi(\cdot)$ and $\gamma(\cdot,\cdot)$ satisfy the following hypothesis of linear growth and Lipschitz condition.

Hypothesis 4.1. *For all $s \geqslant 0$, the noise coefficient $\Phi(\cdot) : \mathrm{H}^s \to \mathscr{L}_2(\mathcal{H}, \mathrm{H}^s)$ and $\gamma(\cdot,\cdot) : \mathrm{H}^s \times Z \to \mathrm{H}^s$ satisfy*

(H.1) *(Growth Condition) For all $\mathbf{u} \in \mathrm{H}^s(\mathbb{R}^n)$ and for all $t \in [0,T]$, there exists a positive constant K such that*

$$\big\|\Phi(\mathbf{u})\big\|_{\mathscr{L}_2(\mathcal{H},\mathrm{H}^s)}^2 + \int_Z \big\|\gamma(\mathbf{u}, z)\big\|_{\mathrm{H}^s}^2 \lambda(\mathrm{d}z) \leqslant K\big(1 + \|\mathbf{u}\|_{\mathrm{H}^s}^2\big).$$

(H.2) *(Lipschitz Condition) For all $t \in [0,T]$ and for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathrm{H}^s(\mathbb{R}^n)$, there exists a positive constant L such that*

$$\big\|\Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_2)\big\|_{\mathscr{L}_2(\mathcal{H},\mathrm{H}^s)}^2 + \int_Z \big\|\gamma(\mathbf{u}_1, z) - \gamma(\mathbf{u}_2, z)\big\|_{\mathrm{H}^s}^2 \lambda(\mathrm{d}z) \leqslant L\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathrm{H}^s}^2.$$

The existence and uniqueness of local strong solutions for the stochastic Euler equations with multiplicative Lévy noise can be proved in the same way as of additive noise case. Proposition 2.11 and Proposition 3.1 can be proved for multiplicative noise case with some changes in the proof due to the presence of $\mathbf{u}(\cdot)$ in the noise coefficient. Similar estimates in Proposition 2.11 can be obtained with the help of growth condition in Hypothesis 4.1 and the Cauchy sequence result in Proposition 3.1 can be obtained with the help of Lipschitz condition in Hypothesis 4.1. In multiplicative noise case, we have to replace Proposition 3.8 and Proposition 3.9 from Section 3, by the following propositions.

Proposition 4.2. *Let τ_N be the stopping time defined in (2.67), then for any $s' > n/2+1$, $s' < s$ and $t \in [0, T \wedge \tau_N)$,*

$$\mathcal{S}_R\bigg(\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg) \to \int_0^t \Phi\big(\mathbf{u}(s)\big)\,\mathrm{dW}(s), \quad \text{as } R \to \infty,$$

in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$.

Proof. By using (2.17), Hypothesis 4.1 and the Burkholder–Davis–Gundy inequality, for $0 < \varepsilon < 1$ and $T > 0$, we get

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg) - \int_0^t \Phi\big(\mathbf{u}(s)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{H}^{s'-1}}^2\bigg] \\
&\quad\leqslant \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg) - \bigg(\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg)\bigg\|_{\mathrm{H}^{s'-1}}^2\bigg] \\
&\qquad + \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \big(\Phi\big(\mathbf{u}^R(s)\big) - \Phi\big(\mathbf{u}(s)\big)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{H}^{s'-1}}^2\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{H}^{s'-1+\varepsilon}}^2\bigg] \\
&\qquad + \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \big(\Phi\big(\mathbf{u}^R(s)\big) - \Phi\big(\mathbf{u}(s)\big)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{H}^{s'}}^2\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \Phi\big(\mathbf{u}^R(s)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{H}^{s}}^2\bigg] \\
&\qquad + \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \mathrm{J}^{s'}\big(\Phi\big(\mathbf{u}^R(s)\big) - \Phi\big(\mathbf{u}(s)\big)\big)\,\mathrm{dW}(s)\bigg\|_{\mathrm{L}^2}^2\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\big\|\Phi\big(\mathbf{u}^R(t)\big)\big\|_{\mathscr{L}_2(\mathcal{H},\mathrm{H}^s)}^2\,\mathrm{d}t\bigg] + \mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\big\|\Phi\big(\mathbf{u}^R(t)\big) - \Phi\big(\mathbf{u}(t)\big)\big\|_{\mathscr{L}_2(\mathcal{H},\mathrm{H}^{s'})}^2\,\mathrm{d}t\bigg] \\
&\quad\leqslant \frac{CK}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\big(1 + \|\mathbf{u}^R(t)\|_{\mathrm{H}^s}^2\big)\,\mathrm{d}t\bigg] + L\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\|\mathbf{u}^R(t) - \mathbf{u}(t)\|_{\mathrm{H}^{s'}}^2\,\mathrm{d}t\bigg] \\
&\quad\leqslant \frac{CKT}{R^\varepsilon}\bigg(1 + \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}^R(t)\|_{\mathrm{H}^s}^2\bigg]\bigg) + LT\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\|\mathbf{u}^R(t) - \mathbf{u}(t)\|_{\mathrm{H}^{s'}}^2\bigg] \\
&\quad\to 0, \quad \text{as } R \to \infty,
\end{aligned}$$

since $\mathbf{u}^R \in \mathrm{L}^2(\Omega; \mathrm{L}^2(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$ for any $s > n/2+1$ (a proposition similar to Proposition 2.11 in multiplicative noise case) and $\mathbf{u}^R \to \mathbf{u} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T\wedge\tau_N; \mathrm{H}^{s'}(\mathbb{R}^n)))$ for any $s' < s$ (a proposition similar to Proposition 3.2 in multiplicative noise case). □

Proposition 4.3. *Let τ_N be the stopping time defined in (2.67), then for any $s' > n/2+1$, $s' < s$ and $t \in [0, T \wedge \tau_N)$,*

$$\mathcal{S}_R\bigg(\int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg) \to \int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \quad \text{as } R \to \infty,$$

in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^{s'-1}(\mathbb{R}^n)))$.

Proof. By using (2.17), Hypothesis 4.1 and the Burkholder–Davis–Gundy inequality, for $0 < \varepsilon < 1$ and $T > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg) - \int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1}}\bigg] \\
&\quad\leqslant \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\mathcal{S}_R\bigg(\int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg) - \int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1}}\bigg] \\
&\qquad+ \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant \tau_N}\bigg\|\int_0^t \int_{\mathrm{Z}} \big(\gamma\big(\mathbf{u}^R(s-), z\big) - \gamma\big(\mathbf{u}(s-), z\big)\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'-1+\varepsilon}}\bigg] \\
&\qquad+ \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant \tau_N}\bigg\|\int_0^t \int_{\mathrm{Z}} \big(\gamma\big(\mathbf{u}^R(s-), z\big) - \gamma\big(\mathbf{u}(s-), z\big)\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s'}}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\bigg\|\int_0^t \int_{\mathrm{Z}} \gamma\big(\mathbf{u}^R(s-), z\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{H}^{s}}\bigg] \\
&\qquad+ \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant \tau_N}\bigg\|\int_0^t \int_{\mathrm{Z}} \mathrm{J}^{s'}\big(\gamma\big(\mathbf{u}^R(s-), z\big) - \gamma\big(\mathbf{u}(s-), z\big)\big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)\bigg\|^2_{\mathrm{L}^2}\bigg] \\
&\quad\leqslant \frac{C}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N} \int_{\mathrm{Z}} \big\|\gamma\big(\mathbf{u}^R(t), z\big)\big\|^2_{\mathrm{H}^s}\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg] \\
&\qquad+ \mathbb{E}\bigg[\int_0^{T\wedge\tau_N} \int_{\mathrm{Z}} \big\|\gamma\big(\mathbf{u}^R(t), z\big) - \gamma\big(\mathbf{u}(t), z\big)\big\|^2_{\mathrm{H}^{s'}}\lambda(\mathrm{d}z)\,\mathrm{d}t\bigg] \\
&\quad\leqslant \frac{CK}{R^\varepsilon}\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\big(1 + \big\|\mathbf{u}^R(t)\big\|^2_{\mathrm{H}^s}\big)\,\mathrm{d}t\bigg] + L\mathbb{E}\bigg[\int_0^{T\wedge\tau_N}\big\|\mathbf{u}^R(t) - \mathbf{u}(t)\big\|^2_{\mathrm{H}^{s'}}\,\mathrm{d}t\bigg] \\
&\quad\leqslant \frac{CKT}{R^\varepsilon}\bigg(1 + \mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R(t)\big\|^2_{\mathrm{H}^s}\bigg]\bigg) + LT\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T\wedge\tau_N}\big\|\mathbf{u}^R(t) - \mathbf{u}(t)\big\|^2_{\mathrm{H}^{s'}}\bigg] \\
&\quad\to 0, \quad \text{as } R \to \infty,
\end{aligned}$$

since $\mathbf{u}^R \in \mathrm{L}^2(\Omega; \mathrm{L}^2(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$ for any $s > n/2+1$ (a proposition similar to Proposition 2.11 in multiplicative noise case) and $\mathbf{u}^R \to \mathbf{u} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^{s'}(\mathbb{R}^n)))$ for any $s' < s$ (a proposition similar to Proposition 3.2 in multiplicative noise case). $\square$

Let us now state the main theorem for incompressible, stochastic Euler equations with multiplicative Lévy noise (see (4.1)–(4.2)).

Theorem 4.4. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a given probability space. Let τ_N be the stopping time defined by*

$$\tau_N := \inf_{t \geqslant 0}\left\{t : \left\|\nabla \mathbf{u}(t)\right\|_{\mathrm{L}^\infty} + \left\|\mathbf{u}(t)\right\|_{\mathrm{H}^{s-1}} \geqslant N\right\}, \tag{4.4}$$

and the noise coefficient satisfy Hypothesis 4.1. Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0 \in \mathrm{L}^2(\Omega; \mathrm{H}^s(\mathbb{R}^n))$, for $s > n/2+1$ be given. Then, there exists a local in time strong solution $(\mathbf{u}, T \wedge \tau_N)$ of the problem (4.1)–(4.2) such that, for any $T > 0$

(i) *the energy estimate*

$$\mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T \wedge \tau_N} \left\|\mathbf{u}(t)\right\|_{\mathrm{H}^s}^2\Big] \leqslant \left(1 + 2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\right]\right) e^{(18K+4N)T} < \infty,$$

(ii) *for a given $0 < \delta < 1$,*

$$\mathbb{P}\{\tau_N > \delta\} \geqslant 1 - C\delta^2\left\{1 + 2\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathrm{H}^s}^2\right]\right\},$$

where C is a positive constant independent of $\mathbf{u}$ and δ,

(iii) $\mathbf{u} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0, T \wedge \tau_N; \mathrm{H}^s(\mathbb{R}^n)))$,

(iv) *the $\mathscr{F}_t$-adapted paths of $(\mathbf{u}, T \wedge \tau_N)$ are càdlàg,*

(v) *the solution $(\mathbf{u}, T \wedge \tau_N)$ is pathwise unique,*

(vi) *there exists a unique maximal local strong solution $(\mathbf{u}, \tau_\infty)$, where $\tau_\infty = \lim_{N\to\infty} \tau_N$.*

Theorem 4.4 can be proved in the same way as of Theorem 3.11.

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